The Class of Constraint Satisfaction Problems over a Knot

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Abstract. This work presents a method for associating a class of constraint satisfaction problems to a three-dimensional knot. Given a knot, one can build a knot quandle, which is a generally an infinite, free algebra. The desired collection of problems is derived from the finite quotients of the knot quandle by applying theory that relates finite algebras to constraint languages. Along the way, notions of tractable and NP-complete quandles and knots are developed. Finally, a partial, computational classification of Rolfsen's Knot Table [26] is undertaken where it is shown that all tricolorable knots in this collection are NP-complete.

1 Introduction

Since Cook's formulation of NP-completeness [4], computer scientists have labored to unravel the mysteries of nondeterministic polynomial time [29]. Early efforts included the building of a catalogue of individual NP-complete combinatorial problems in the hope that one or more would provide significant insight [19]. In the meantime, more structurally-oriented approaches have emerged that instead focus on subclasses of NP. A notable example is the development of descriptive complexity [7, 13], which considers complexity classes axiomatized by some fragment of (existential) second-order logic.

Another promising avenue restricts attention to subclasses of CSP, the class of constraint satisfaction problems [22]. Early on, Schaefer proved that every Boolean constraint satisfaction problem is NP-complete or tractable [27]. Feder and Vardi conjectured that this dichotomy holds for all of CSP [8]. Since then, Bulatov has extended Schaefer's result to three-element domains [2].

More importantly, Feder and Vardi showed that a solution to a constraint satisfaction problem corresponds to a homomorphism between certain finite, first-order structures. This idea was further refined by Jeavons and others [15, 16], and has led to significant insight into the structure of tractable subclasses of CSP [10, 11]. In particular, Jeavons, Cohen, and Pearson explored the relationship between CSP and universal algebra [17].

In [3], Bulatov, Jeavons, and Krokhin used the language of relational clones [30] and tame congruence theory [12, 21] to formulate notions of tractable and NP-complete algebras. They showed that the process of classifying finite algebras as tractable or NP-complete need only consider the surjective ones, and proved P/NP-complete dichotomy for finite strictly simple surjective algebras. Moreover, they identified the class of idempotent algebras, all of which are surjective, as a prime target for the next round of dichotomy results.

This article introduces a new geometric dimension to the study of the constraint satisfaction problem, tractability, and NP-completeness. In particular, it presents a notion of constraint satisfaction problem over a 3-dimensional knot [5, 26]. This is made possible by the relationship between knots and a class of algebras known as quandles [18]. Since each quandle axiom is an identity, quandles form a variety [12, 21]. Quandles are idempotent, so their study is relevant to current research into CSP.

2 Knots and Quandles

The basics of knot theory are reviewed in this section. More extensive treatment can be found in [5, 26]. A **knot** $\mathcal{K}$ is an injective, continuous, open mapping from the unit circle $\mathbb{S}^1$ into $\mathbb{R}^3$. $\mathcal{K}$ is identified with its oriented image in $\mathbb{R}^3$, which is homeomorphic to $\mathbb{S}^1$. Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are **ambient isotopic** if the complement spaces $\mathbb{R}^3 - \mathcal{K}_1$ and $\mathbb{R}^3 - \mathcal{K}_2$ are homeomorphic. This captures the notion of continuous deformation of one knot into another. That is, $\mathcal{K}_1$ can be so transformed into $\mathcal{K}_2$ if and only if the two knots are ambient isotopic.

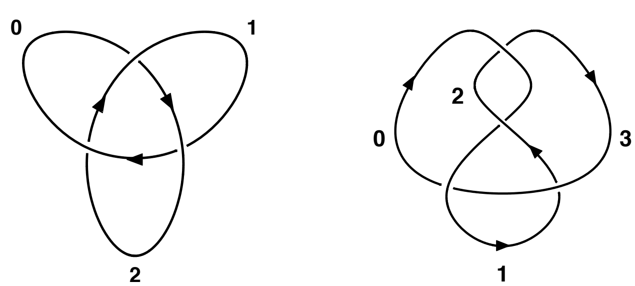


Fig. 1. Trefoil (3_1) and Figure Eight (4_1) Knots

It is often convenient to visualize a knot via 2-dimensional projection. The Trefoil and Figure Eight knots are so presented in Figure 1. The arrows indicate the orientation. Notice that this projection has three crossings. Any knot that is ambient isotopic to the Trefoil will have at least three crossings in all of its projections. A knot projection with this minimal property is called **reduced**.

For the duration of this article, the portion of a knot between successive crossings is called an **arc**. Each arc is labeled by an integer.

Certain knots $\mathcal{K}$ have an **Alexander-Briggs** representation n_k. In this case, $\mathcal{K}$ has rank k among all represented knots that have a reduced projection with n crossings. The relative rank k is purely nominal. The Alexander-Briggs notation for the Trefoil is 3_1.

2.1 Quandles

Joyce [18] introduced the notion of quandle as an algebraic invariant of knots. Quandles are defined here from the point of view of universal algebra [12, 21], since this perspective is useful to the development of Section 2.3.

Definition 1. *A quandle* $(Q, \triangleright, \blacktriangleright)$ *is a set* Q *together with binary operations* $\triangleright, \blacktriangleright \colon Q \times Q \to Q$ *satisfying the following axioms.*

Idempotence: $\forall x(x \triangleright x = x)$
Right Cancellation A: $\forall xy((x \triangleright y) \blacktriangleright y = x)$
Right Cancellation B: $\forall xy((x \blacktriangleright y) \triangleright y = x)$
Right Self-Distributivity: $\forall xyz((x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z))$

The simplest examples are the **unary quandles** U_n where n is a positive integer. The underlying set of U_n is $\{0, 1, \ldots, n-1\}$ and $\triangleright$ and $\blacktriangleright$ simply project the first argument: $x \triangleright y = x \blacktriangleright y = x$. That U_n satisfies the quandle identities is immediately obvious. The **Dihedral quandle** D_n has the same underlying

$\triangleright, \blacktriangleright$	0	1
0	0	0
1	1	1

$\triangleright, \blacktriangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Table 1. U_2 and D_3

set as U_n but its operations are defined by $x \triangleright y = x \blacktriangleright y = 2y - x$, where the arithmetic occurs modulo n.

2.2 The Knot Quandle

Given a projection for a knot $\mathcal{K}$, one can construct a **quandle presentation** as follows. To each crossing, assign a simple identity using the relevant arc labels and one of two binary operations, $\triangleright$ and $\blacktriangleright$. Figure 2 illustrates the two cases. In the left diagram, arc a passes to arc c at the point where arc b crosses to the left. This translates to the equation $c = a \triangleright b$. The right diagram of Figure 2 has b crossing to the right instead, which corresponds to $c = a \blacktriangleright b$.

For example, the Trefoil projection of Figure 1 has the following presentation:

$$\mathcal{Q}(3_1) = \langle 0, 1, 2 | 1 = 0 \triangleright 2, 2 = 1 \triangleright 0, 0 = 2 \triangleright 1 \rangle.$$

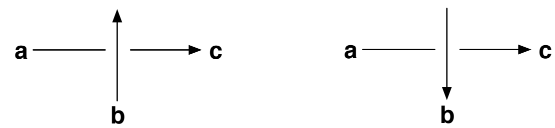


Fig. 2. Left and Right Crossings

2.3 The Reidemeister Moves

The relevance of quandle structure to knots can be inferred from the **Reidemeister Moves**. Reidemeister [25] proved that ambient isotopy can proceed through successive applications of three types of transformations. In each move, some portion of the knot is the focus. If that part resembles one of the two diagrams of the move, it may be transformed to resemble the other diagram. It is assumed that the rest of the knot remains unchanged during this deformation. This results in a knot that is ambient isotopic to the previous one.

An example of **Type I** move appears in Figure 3. The left hand diagram has

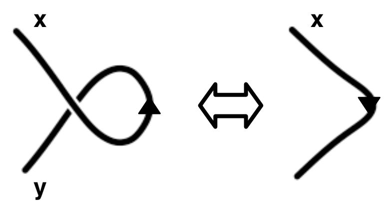


Fig. 3. Type I Reidemeister Move

a segment of the knot looping behind itself. The crossing forms two arcs, x and y, and thus corresponds to the equation $y = x \triangleright x$. A simple twist of the loop yields the right hand diagram, reducing this part to one arc x. Here the role of y within the rest of the knot is now fulfilled by x. Hence $x = y = x \triangleright x$, so from ambient isotopy, one may infer that $\triangleright$ is idempotent.

Figure 4 presents an instance of a **Type II** move. The diagram on the left has arc y crossing over two points of the knot in succession, while on the right, y has been placed so that these two crossings do not occur. The point on the left hand diagram labeled by w is equated with its analogous location in the other diagram. Thus, $x = w = z \blacktriangleright y = (x \triangleright y) \blacktriangleright y$. Reversing the orientation on arc y leads to the companion right cancellation identity.

Lastly, right self distributivity can be gleaned from a **Type III** move (Figure 5). In this scenario, there are two segments that form one crossing in the center of both diagrams, and a third, single-arc segment z that crosses over the other two segments. The diagrams differ as to whether z crosses to the left or right

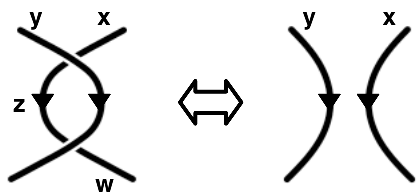


Fig. 4. Type II Reidemeister Move

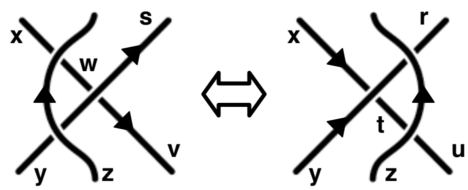


Fig. 5. Type III Reidemeister Move

of the central crossing. From Move I, one can infer that $v = u$. Analyses of the crossings in both diagrams yield

$$(x \rhd y) \rhd z = t \rhd z = u = v = w \rhd s = (x \rhd z) \rhd (y \rhd z).$$

The quandle axioms guarantee that ambient isotopic knots, as well as different projections of the same knot, have isomorphic knot quandles. Hence, the functorial notation $\mathcal{Q}(\mathcal{K})$ of Section 2.2 is well defined.

Since right cancellation ensure that the equation $x \rhd y = z$ is provably equivalent to $x = z \blacktriangleright y$, the operation $\blacktriangleright$ is uniquely determined by $\rhd$. One may dispense entirely with $\blacktriangleright$. For example, Q' is a subquandle of a quandle Q if it is closed under $\rhd$, and a function $h : Q \to Q''$ is a quandle homomorphism if it preserves $\rhd$. Henceforth, finite quandles will be presented via the Cayley table for $\rhd$ alone. Elimination of $\blacktriangleright$ also extends to knot quandles: The knot quandle of 4_1 (Figure 1), which has both types of crossings, can be expressed as

$$\mathcal{Q}(4_1) = \langle 0, 1, 2, 3 | 0 = 1 \rhd 2, 2 = 1 \rhd 3, 2 = 3 \rhd 0, 0 = 3 \rhd 1 \rangle.$$

2.4 Tricolorable Knots and Finite Images

A precursor to Joyce's concept of quandle is **tricolorability** [24]. A tricoloring of a knot $\mathcal{K}$ is an assignment of one of three colors $\{0, 1, 2\}$ to each arc of $\mathcal{K}$ in such a way that every crossing either has three stands of the same color or one arc of each color, and such that at least two colors are employed.

The integer labels in Figure 1 constitute a tricoloring of 3_1. One may also view these labels as elements of D_3 (Table 1). Moreover, the equations of $\mathcal{Q}(3_1)$ hold in D_3. In general, a tricoloring of $\mathcal{K}$ corresponds to a surjective quandle homomorphism $h : \mathcal{Q}(\mathcal{K}) \to D_3$. In other words, $\mathcal{K}$ is tricolorable if and only if D_3 is a quotient of $\mathcal{Q}(\mathcal{K})$.

Important to the development of Section 4 are situations in which there is a knot $\mathcal{K}$, a finite quandle Q, and a surjective quandle homomorphism $h : \mathcal{Q}(\mathcal{K}) \to Q$. For the sake of notational convenience, such a quotient Q of $\mathcal{Q}(\mathcal{K})$ will be called a **$\mathcal{K}$-quandle**. For example, it was shown above that D_3 is a 3_1-quandle.

3 Constraint Satisfaction Problems over Finite Quandles

The basic elements of a **constraint satisfaction problem** [22] include a finite domain A, a countable collection of variables $X = \{v_1, v_2, \ldots, v_n, \ldots\}$, and a **constraint language** Γ, which is a collection of relations $R \subseteq A^n$ for various n. In this context, a **constraint over Γ** is a pair $\langle (v_{i_1}, v_{i_2}, \ldots, v_{i_m}), R\rangle$, where R is a relation in Γ of arity m.

3.1 Example: 2-Colorable Graphs

The main concepts of constraint satisfaction are illustrated through **graph 2-coloring**, a well known NP-complete constraint satisfaction problem [1, 9]. A graph $G = (V, E)$ is 2-colorable if one can assign to each vertex $v \in V$ one of two colors, say 0 and 1, in such a way that for each edge $(v, v') \in E$, v and v' are assigned different colors. 2-COLOR is the problem of determining whether a given finite graph is 2-colorable. For example, consider the graphs of Figure

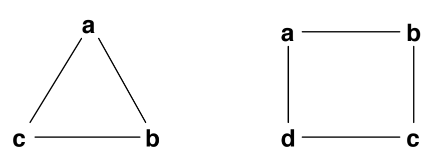
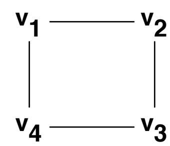


Fig. 6. C_3 and C_4

6, the cyclic graphs with three and four nodes. C_4 is 2-colorable, as witnessed by coloring nodes a and c with 0 and b and d with 1. On the other hand, the topology of C_3 ensures that no two of its vertices may have the same color. Consequently, C_3 is not 2-colorable.

One can recast 2-colorability within the realm of formal constraints. First, let $A = \{0, 1\}$ and label the vertices of G with the first $k = |V|$ variables of X: $V = \{v_1, \ldots, v_k\}$. For each edge $(v_i, v_j) \in E$, construct a constraint $\langle (v_i, v_j), \Delta^c\rangle$, where $\Delta^c = \{(0, 1), (1, 0)\}$. Here $\Gamma = \{\Delta^c\}$. For instance, Figure 7 features C_4 labeled by variables of X along with the associated set of constraints.



$$\mathcal{C}_4 = \{\langle (v_1, v_2), \Delta^c \rangle, \langle (v_2, v_3), \Delta^c \rangle, \langle (v_3, v_4), \Delta^c \rangle, \langle (v_4, v_1), \Delta^c \rangle\}$$

Fig. 7. C_4 with Constraints

3.2 CSP(Γ)

Definition 2. *Given a domain A, a collection of variables X, and a constraint language Γ over A,* CSP(Γ) *is the combinatorial decision problem with the following components.*

Instance: *An instance of* CSP(Γ) *is a triple $\mathcal{I} = (X', A, \mathcal{C})$ where $\mathcal{C}$ is a finite set of constraints over Γ and X' is the finite subset of variables in X that appear in $\mathcal{C}$.*

Solution: *A solution to an instance $\mathcal{I}$ of* CSP(Γ) *is a function $\theta : X' \to A$ such that for every constraint $\langle (v_1, v_2, \ldots, v_m), R\rangle \in \mathcal{C}$,*

$$(\theta(v_1), \theta(v_2), \ldots, \theta(v_m)) \in R.$$

Letting $A = \{0, 1\}$, $V_4 = \{v_1, v_2, v_3, v_4\}$, and $\mathcal{C}_4$ equal the set of constraints of Figure 7, the formal instance of CSP($\{\Delta^c\}$) corresponding to C_4 is

$$\mathcal{I}_4 = (V_4, A, \mathcal{C}_4).$$

The 2-coloring θ_4 that sends v_1, v_3 to 0 and v_2, v_4 to 1 is a solution for $\mathcal{I}_4$. On the other hand the instance $\mathcal{I}_3$ associated to C_3 has no solution, since C_3 is not 2-colorable.

For a finite constraint language Γ, one can formulate a simple algorithm that, when given an instance $\mathcal{I} = (X', A, \mathcal{C})$ of CSP(Γ) and $\theta : X' \to A$, verifies whether or not θ is a solution for $\mathcal{I}$ in time polynomial in the length of the description of $\mathcal{I}$. Consequently, CSP(Γ) is solvable in nondeterministic polynomial time for all finite CSP(Γ).

Definition 3. *The constraint language Γ is* **tractable** *if for every finite $\Gamma' \subseteq \Gamma$ there exists a polynomial time algorithm to decide* CSP(Γ'). *Γ is* **NP-complete** *if* CSP(Γ') *is NP-complete for some finite $\Gamma' \subseteq \Gamma$.*

3.3 CSP(Q)

According to the template of Definition 2, one only need find suitable representatives for the domain A and the constraint language Γ in order to construct a notion of constraint satisfaction problem over a finite quandle Q. The former is straightforward; let $A = Q$. For the latter, one may appeal to the subquandle structure of Q. A **subpower** of Q is a subquandle of the direct product Q^n for some nonnegative integer n. Denote the set of subpowers of Q by $\mathbf{Sub}(Q)$. Notice that each $Q' \in \mathrm{Sub}(Q)$ is a relation over Q. Let $\Gamma = \mathrm{Sub}(Q)$.

Definition 4. *Q is* **tractable** *if* $\mathrm{Sub}(Q)$ *is tractable. Q is* **NP-complete** *if* $\mathrm{Sub}(Q)$ *is NP-complete. Let* $\mathbf{CSP(Q)}$ *stand for* $\mathrm{CSP}(\Gamma)$ *where* $\Gamma = \mathrm{Sub}(Q)$.

3.4 NP-Complete Quandles

Suppose $Q' \in \mathrm{Sub}(Q)$. Then $\mathrm{Sub}(Q') \subseteq \mathrm{Sub}(Q)$. If Q' is also NP-complete, then there exists a finite $\Gamma \subseteq \mathrm{Sub}(Q')$ with $\mathrm{CSP}(\Gamma)$ NP-complete. Clearly, $\Gamma \subseteq \mathrm{Sub}(Q)$, as well, so Q is NP-complete.

Proposition 1. *If $Q' \in \mathrm{Sub}(Q)$ is NP-complete, then so is Q.*

The unary quandle U_2 of Table 1 plays a central role in this article. Every relation over $\{0, 1\}$ is a subpower of U_2, including Δ^c of Section 3.1. Since Δ^c is NP-complete, so is U_2. Idempotence and right cancellation alone dictate that U_2 is the only quandle of size 2, up to isomorphism. This proves the following.

Corollary 1. *Suppose Q has a subquandle Q' of size 2. Then Q is NP-complete.*

$\rhd$	0	1	2	3	4	5
0	0	3	1	4	2	0
1	2	1	5	0	1	3
2	4	0	2	2	5	1
3	1	5	3	3	0	4
4	3	4	0	5	4	2
5	5	2	4	1	3	5

Table 2. Wood_6

The vast majority of finite quandles of size $n > 1$ have a subquandle isomorphic to U_2, and so are NP-complete. Included in this class is the quandle Wood_6 of Table 2. Notice that it has $\{0, 5\}$ as a subquandle. Also, Wood_6 is a 3_1-quandle since it satisfies the equations of $\mathcal{Q}(3_1)$ and is generated by $\{0, 1, 2\}$. This quandle is used in Section 4.1 for the computational classification of knots.

3.5 Strictly Simple Quandles and Tractability

Let $\mathcal{A} = (A, F)$ be an algebra. A **term operation** is a constant-free expression $e(x_1, x_2, \ldots, x_k)$ over F regarded as a function $e : A^k \to A$. $\mathcal{A}$ is **surjective** if all of its term operations are surjective. For example, idempotence ensures that all quandles are surjective. $\mathcal{A}$ is **strictly simple** if it has no nontrivial subalgebras or quotients.

In [3, 6], Schaefer's dichotomy result is extended to finite strictly simple surjective algebras. It is also shown that the NP-complete, strictly simple surjective algebras must have only **essentially unary** operations. A binary operation $* : A^2 \to A$ is essentially unary if there exists a unary operator $u : A \to A$ and an index i such that $x_1 * x_2 = u(x_i)$ for all values of $x_1, x_2 \in A$. The only strictly simple quandle with an essentially unary $\triangleright$ is the unary quandle U_2.

Thus, any other strictly simple quandle must be tractable. It can be shown that for a prime $p > 2$, D_p is strictly simple. Hence, there are infinitely many tractable quandles.

3.6 Notes for Section 3

The formulation of the CSP has experienced some evolution since its introduction by Montanari [20, 22]. The presentation of CSP(Γ) here closely follows the notational conventions of [3, 6]. In fact, Definition 2 is lifted from this work virtually unaltered.

The presentation of CSP(Q) of Sections 3.3 through 3.5 is just the tip of the algebra/CSP iceberg [3, 6, 17, 8]. In this tradition, given a finite algebra $\mathcal{A} = (A, F)$, the constraint language of interest is the set $\mathbf{Inv}(\mathcal{A})$ of relations over A fixed by the non constant operations in F. For a quandle Q, $\mathrm{Inv}(Q) = \mathrm{Sub}(Q)$, which significantly simplifies the presentation.

4 Constraint Satisfaction Problems over Knots

Given a knot $\mathcal{K}$, the knot quandle $\mathcal{Q}(\mathcal{K})$ is generally a (countably) infinite algebra, and so does not present an ideal setting for constraint satisfaction problems as formulated in Section 3. A more appropriate context can be had by instead considering a finite, homomorphic image of $\mathcal{Q}(\mathcal{K})$ – i.e. a $\mathcal{K}$-quandle.

In this way, a **constraint satisfaction problem over** $\mathcal{K}$ is a constraint satisfaction problem over Q for some $\mathcal{K}$-quandle Q. $\mathcal{K}$ is **tractable** if Q is tractable for all $\mathcal{K}$-quandles Q, and $\mathcal{K}$ is **NP-complete** if Q is NP-complete for at least one $\mathcal{K}$-quandle Q.

4.1 A Partial Computational Classification of Rolfsen's Knot Table

The Rolfsen Knot Table [26] depicts certain knots whose reduced forms have 10 or fewer crossings. In this section, for each knot $\mathcal{K}$ it is determined whether Wood_6, which is shown to be NP-complete in Section 3.4, is a $\mathcal{K}$-quandle. An affirmative answer for $\mathcal{K}$ proves that $\mathcal{K}$ is NP-complete. For example, in Section 3.4 it was also determined that Wood_6 is a 3_1-quandle, so 3_1 is NP-complete.

The function $g : \mathrm{Wood}_6 \to D_3$ defined in Figure 8 is a quandle homomorphism. So if $h : \mathcal{Q}(\mathcal{K}) \to \mathrm{Wood}_6$ is a surjective homomorphism, then so is $g \circ h : \mathcal{Q}(\mathcal{K}) \to D_3$.

Subsequently, only the tricolorable knots in Rolfsen's collection need to be tested (see Table 3 of the Appendix). A program written in SWI-Prolog [31]

$$g(x) = \begin{cases} 0 & x = 0, 5 \\ 1 & x = 1, 4 \\ 2 & x = 2, 3 \end{cases} \tag{1}$$

Fig. 8. $g : \mathrm{Wood}_6 \to D_3$

converted the braid representation of each of these knots to a quandle presentation and then searched for a nontrivial solution for the presentation in Wood_6. For good measure, the latter stage of this process was repeated using alternative quandle presentations computed by hand from KnotPlot [28] images. All solutions were verified by paper and pencil. Every knot in Table 3 was proved NP-complete this way. The results for 8 or fewer crossings appear in the Appendix.

4.2 Current Challenges

Conspicuously absent from Section 4.1 is any mention of tractable knots. So far, verifying tractability has proved substantially more challenging than demonstrating NP-completeness. At this time, only the Unknot 0_1, which has trivial knot quandle, is known to be tractable. This remains an active area of research for the ASC lab.

Meanwhile, Corollary 1 ensures a plethora of NP-complete quandles, but only two are known to serve as $\mathcal{K}$-quandles, and they appear to classify precisely the same knots. Further progress in this area will require a larger catalogue of NP-complete $\mathcal{K}$-quandles. Possible avenues of exploration include finite Alexander quandles [14, 23].

5 Conclusion

The theory developed in Sections 3 and 4 might provide a useful classifying invariant for knots. However, this was not the original purpose of this work. Rather, the motivation has been to provide a path whereby the tools of knot theory can shed light on the mysteries of nondeterministic polynomial time through the constraint satisfaction problem. The authors of this article continue to seek geometric characteristics of knots, such as tricolorability, that are in some way related to computational phenomena.

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Appendix: Rolfsen's Tricolorable Knots and Wood$_6$

Crossings	Rank
3	1
6	1
7	4, 7
8	5, 10, 11, 15, 18, 19, 20, 21
9	1, 2, 4, 6, 10, 11, 15, 16,
	17, 23, 24, 28, 29, 34, 35, 37,
	38, 40, 46, 47, 48
10	4, 5, 9, 10, 14, 19, 21, 29,
	31, 32, 36, 40, 42, 59, 61, 62,
	63, 64, 65, 66, 67, 68, 69, 74,
	75, 76, 77, 78, 82, 84, 85, 87,
	89, 96, 97, 98, 98, 99, 103, 106,
	107, 108, 112, 113, 120, 122, 136, 139,
	140, 141, 142, 143, 144, 145, 146, 147,
	158, 159, 160, 163, 164, 165

Table 3. Tricolorable Knots of the Rolfsen Knot Table