Function Fields with Class Number Indivisible by a prime ℓ

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Definitions: Number Field & Ring of Integers

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but 2, 3, and $\sqrt{-6}$ are irreducible in $\mathbb{Z}[\sqrt{-6}]$.

Thus, $\mathbb{Z}[\sqrt{-6}]$ is not a UFD.

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Note $\langle -6 \rangle = \langle -2 \rangle \langle 3 \rangle = \langle \sqrt{-6} \rangle^2 = \langle 2, \sqrt{-6} \rangle^2 \langle 3, \sqrt{-6} \rangle^2$.

Equivalence Relation

Nonzero ideals $I \sim J$ if $aI = bJ$ for some nonzero $a, b \in \mathcal{O}_K$.

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- Associativity: ✓
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- Inverses: hard

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Theorem

For Dedekind domains, UFD $\Leftrightarrow$ PID.

- Thus, $\mathcal{O}_K$ is a UFD if and only if $h_K = 1$.
- Roughly, the class number measures the closeness of $\mathcal{O}_K$ to being a UFD.

Class Numbers of Quadratic Fields:

d	2	3	5	6	7	10	11	13	14	15	17	19	21
$\mathrm{Cl}_{\mathbb{Q}(\sqrt{d})}$	1	1	1	1	1	2	1	1	1	2	1	1	1
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The class number of $\mathbb{Q}(\sqrt{d})$, $d < 0$, *is 1 if and only if* $d = -1, -2, -3, -7, -11, -19, -43, -67$ *or* -163.

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Open Question

Are there infinitely many real quadratic number fields with class number one?

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- Note that $\mathbb{F}(T)$ is the field of fractions of polynomials in T over $\mathbb{F}$.
- We can define the ring of integers of K in the same way as for number fields.
- The ring of integers of $\mathbb{F}(T)$ is $\mathbb{F}[T]$, the ring of polynomials in T over $\mathbb{F}$.

Number Fields vs. Function Fields

Number Field	Function Field
$\begin{array}{ccc} \mathcal{O}_K & \subset & K \\ \mid & & \mid \\ \mathbb{Z} & \subset & \mathbb{Q} \end{array}$	$\begin{array}{ccc} \mathcal{O}_K & \subset & K \\ \mid & & \mid \\ \mathbb{F}_q[T] & \subset & \mathbb{F}_q(T) \end{array}$

	$\mathbb{Z}$	$\mathbb{F}_q[T]$
UFD	yes	yes
irreducibles	(infinitely many) primes	(infinitely many) irreducible polynomials
units	$\{\pm 1\}$ (finitely many)	$\mathbb{F}_q^\times$ (finitely many)
residue class	$\lvert\mathbb{Z}/n\mathbb{Z}\rvert = \lvert n\rvert$	$\lvert\mathbb{F}_q[T]/f\,\mathbb{F}_q[T]\rvert = q^{\deg f}$

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- The abc-conjecture has been proven!
- Riemann Hypothesis analogue for function fields also proven!
- Every function field is isomorphic to a non-singular projective curve, so we can compute the genus of the function field.
- Still, questions on class numbers of function fields are VERY HARD.

The Case $\ell = 3$

Theorem (Pacelli, Rosen)

Let m be any positive integer, $m > 1$ and $3 \nmid m$. There are a positive density of primes (and prime powers) q such that for a given rational function field $\mathbb{F}_q(T)$, there are infinitely many function fields of degree m over $\mathbb{F}_q(T)$ with divisor class number indivisible by 3.

The Case $\ell = 5$

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Let ζ be a root of the polynomial $X^4 + X^3 + X^2 + X + 1 \in \mathbb{F}_q[X]$, and assume the following conditions on q are true:

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- if $4|m$, then $\gamma + 5\zeta \notin -4\mathbb{F}_q(\zeta)^4$

Constructing the Fields

The Recursion Relation

Define $X_0 = T$ and

$$X_j = \frac{X_{j-1}^5 - 10X_{j-1}^3 + 10\omega X_{j-1}^2 + 5\omega X_{j-1} - 1}{5X_{j-1}(X_{j-1}^3 - 2\omega X_{j-1}^2 - 2\omega X_{j-1} + 1)},$$

for $j \geq 1$ and $\omega \in \mathbb{F}_q(T)$ such that $\omega^2 + \omega - 1 = 0$.

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The Field of Degree m

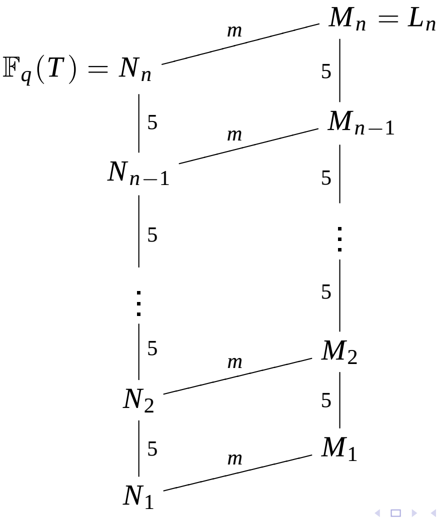
Fix $n \geq 1$. For $1 \leq i \leq n$, define

$$N_i = \mathbb{F}_q(X_{n-i})$$
$$M_i = \mathbb{F}_q(X_{n-i}, \sqrt[m]{5X_n + \gamma}).$$

Let $L_n = \mathbb{F}_q(T)(\sqrt[m]{5X_n + \gamma}) = M_n$.

Field Diagram

$$N_i = \mathbb{F}_q(X_{n-i}) \text{ and } M_i = \mathbb{F}_q(X_{n-i}, \sqrt[m]{5X_n + \gamma})$$



Definition of f_p's

Let p be a divisor of m such that either p is prime or $p = 4$. Define

$$f_p(x) = 5 \sum_{\substack{i=0 \\ i\equiv 0(5)}}^{p} \binom{p}{i} x^{p-i} - \gamma \sum_{\substack{i=0 \\ i\equiv 1(5)}}^{p} \binom{p}{i} x^{p-i} - (5+\gamma\omega) \sum_{\substack{i=0 \\ i\equiv 2(5)}}^{p} \binom{p}{i} x^{p-i}$$

$$+ \omega(\gamma - 5) \sum_{\substack{i=0 \\ i\equiv 3(5)}}^{p} \binom{p}{i} x^{p-i} + (\gamma + 5\omega) \sum_{\substack{i=0 \\ i\equiv 4(5)}}^{p} \binom{p}{i} x^{p-i}$$

$$f_4(x) = x^4 - \frac{4}{5}\gamma x^3 - (\frac{6}{5}\gamma\omega + 6)x^2 + 4\omega(\frac{1}{5}\gamma - 1)x + (\omega + \frac{\gamma}{5})$$

Fact: Each $f_p(x)$ is Eisenstein with respect to the chosen prime $\mathfrak{p} \subset \mathbb{Q}(\omega)$ lying over p, and thus each $f_p(x)$ is irreducible over $\mathbb{Q}(\omega)$.

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Let G be a group acting on a finite set X with cardinality n. If $n \geq 2$ and G acts transitively on X, then there is an element $g \in G$ which acts on X without a fixed point.

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Theorem (Frobenius)

Let f be an irreducible polynomial over $\mathbb{Q}(\omega)$ with Galois group G. The density of primes q for which f has no roots mod q exists, and is equal to $1/|G|$ times the number of $\sigma \in G$ with no fixed points.

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  So, in order to use the same techniques in general, we need a polynomial that generates cyclic extensions of degree ℓ.
- Rikuna showed that the splitting field of the following polynomial has Galois group $\mathbb{Z}/\ell\mathbb{Z}$ over $k(T)$ for certain fields k:

$$\frac{\zeta^{-1}(X-\zeta)^\ell - \zeta(X-\zeta^{-1})^\ell}{\zeta^{-1}-\zeta} - T\,\frac{(X-\zeta)^\ell - (X-\zeta^{-1})^\ell}{\zeta^{-1}-\zeta}.$$

The General Case

Theorem

Let ℓ be a prime and $m > 1$ be any positive integer such that $\ell \nmid m$. Then there are a positive density of primes (and prime powers) q such that for a given rational function field $\mathbb{F}_q(T)$, there are infinitely many function fields of degree m over $\mathbb{F}_q(T)$ with divisor class number indivisible by ℓ.

Conditions on q

Let ζ be a root of $g(X) = X^{\ell-1} + X^{\ell-2} + \cdots + X + 1$ and let $h(X)$ be the minimal polynomial of $\omega = \zeta + \zeta^{-1}$.

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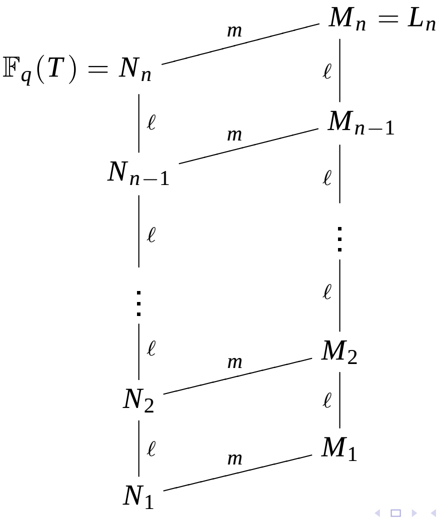

Proving there are infinitely many q

- Recall: We want $\ell\zeta + \gamma \notin \mathbb{F}_q(\zeta)^p$ for all p dividing m and $\ell\zeta + \gamma \notin -4\mathbb{F}_q(\zeta)^4$ if $4 \mid m$.

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- For p a prime or $p = 4$, define a polynomial $f_p(X) \in \mathbb{Q}(\omega)[X]$ as follows:

$$f_p(X) = \sum_{j=0}^{\ell-1} \sum_{\substack{i=0 \\ i \equiv j}}^{p} \binom{p}{i} (a_j\gamma + a_{j-1}\ell) X^{p-i}$$

where $a_j = (\zeta^j - \zeta^{-j})/(\zeta - \zeta^{-1})$ and γ is chosen to make f_p Eisenstein for each $p \mid m$.

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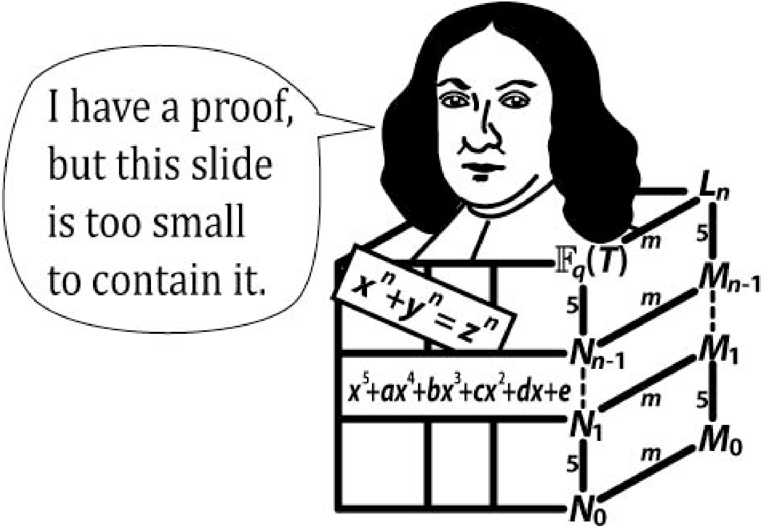
- The remainder of the proof is identical to the $\ell = 5$ case.

Thank You


I have a proof, but this slide is too small to contain it.
$x^n + y^n = z^n$
$x^5+ax^4+bx^3+cx^2+dx+e$
$\mathbb{F}_q(T)$
L_n
M_{n-1}
M_1
M_0
N_{n-1}
N_1
N_0
5
m

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- Tools used in the function field case are unavailable in the number field case, such as the genus of a curve and the Riemann-Hurwitz equation.