Gassmann Equivalent Dessins

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Dessins d'enfants

Definition

A *bicolored dessin d'enfant*, or *dessin* for short, is a pair of permutations $\sigma_0,\ \sigma_1$ of n objects called edges.

- To each cycle in σ_0 draw a white vertex sprouting whiskers labelled counterclockwise by the elements in that cycle.
- Do the same for each cycle in σ_1 drawing black vertices.
- Connect the whiskers with the same label to produce an edge connecting a black vertex to a white vertex.
- We obtain a bicolored (bipartite) graph with a counterclockwise cyclic ordering of the edges at each vertex.

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Example

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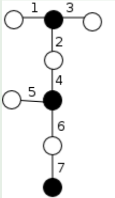
G=simple group of order 168. $H \leq G$ a subgroup of index 7.
Magma gives us the following presentation of G:
$G =< x, y, z : x^2 = 1,\ y^3 = 1,\ z = xy,\ z^7 = 1 >$

The action of x on the 7 cosets G/H gives the permutation $\sigma_0 = (1)(2\ 4)(3)(5)(6\ 7)$ and similarly the action of y on G/H gives the permutation $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$
The permutations $\sigma_0,\ \sigma_1$ give rise to the following dessin:



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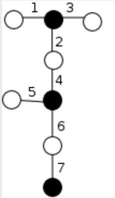

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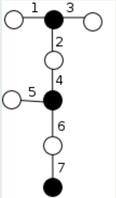

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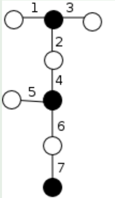
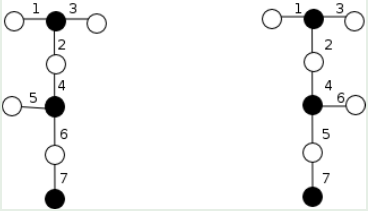


Example

This group G has another subgroup H' of index 7 with H' not conjugate to H. The actions of x and y on G/H' give two permutations $\sigma_0' = (1)(2\ 4)(3)(5\ 7)(6)$ and $\sigma_1' = (1\ 2\ 3)(4\ 5\ 6)(7)$.
This gives a second dessin. The two dessins together are:



$$\sigma_0 = (1)(2\ 4)(3)(5)(6\ 7) \qquad \sigma_0' = (1)(2\ 4)(3)(5\ 7)(6)$$
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Gassmann Triples

Definition

Let G be a group and let H and H' be two subgroups of G. (G, H, H') is a Gassmann triple if for each $g \in G$ the number of fixed points of g acting on G/H is the same as the number of fixed points of g acting on G/H'.

Definition

Gassmann equivalent dessins are bipartite dessins $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ that arise from two chosen elements g_0 and g_1 in G acting by left multiplication on the coset spaces G/H and G/H' where (G, H, H') is a Gassmann triple.

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Branching data

Definition

Branching data for a dessin $\mathbb{D}$ given by $\sigma_0,\ \sigma_1$ is the triple of cycle structures

$$(c.s.(\sigma_0),\ c.s.(\sigma_1),\ c.s.(\sigma_1^{-1}\sigma_0^{-1})).$$

Theorem

If (G, H, H') is a Gassmann triple, then the branching data for $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ coincide.

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Components

Theorem

- *The number of components of a dessin is the number of orbits of* $< \sigma_0, \sigma_1 >$.
- *Gassmann equivalent dessins have the same number of components.*

Genera

Definition

The genus of a connected dessin $\mathbb{D}$ is the number γ given by

$$2n - \sum(\text{length}(c) - 1) = 2 - 2\gamma$$

where n=# of edges and c runs over all cycles in $\sigma_0,\ \sigma_1,$ and $\sigma_1^{-1}\sigma_0^{-1}$.

Theorem

If (G, H, H') is a Gassmann triple, then $\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \gamma_i'$ where k is the number of components of $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$, γ_i denotes the genus of the i-th component of $\mathbb{D}(G/H, g_0, g_1)$ and γ_i' denotes the genus of the i-th component of $\mathbb{D}(G/H', g_0, g_1)$

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Genera

Genera of disconnected Gassmann Equivalent Dessins

Question: Do the individual genera of the components match?

NO!

Example

$G = GL_2(\mathbb{F}_5)$;

$H = \{\begin{pmatrix} a^2 & x \\ 0 & c \end{pmatrix} \mid a^2c \neq 0\}$ and $H' = \{\begin{pmatrix} c & x \\ 0 & a^2 \end{pmatrix} \mid a^2c \neq 0\}$.

(G, H, H') is a Gassmann triple of index 12.

Let $g_0 = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}$ and $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

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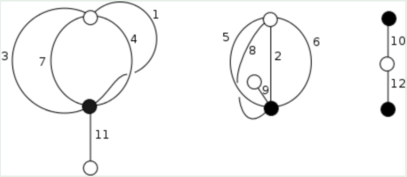
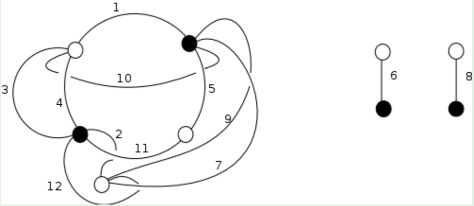
$\mathbb{D}(G/H, g_0, g_1)$

$\gamma_1 + \gamma_2 + \gamma_3 =$
$1 + 1 + 0 = 2$



$\mathbb{D}(G/H', g_0, g_1)$

$\gamma'_1 + \gamma'_2 + \gamma'_3 =$
$2 + 0 + 0 = 2$



Open Questions

- What happens in the cases in which G has an outer automorphism that takes H to H'?
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