Lie Algebras over Finite Fields

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Definition of Lie Algebra

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Let L be an algebra over a field k. Then L is called a **Lie algebra** over k if there exists a bilinear map

$$[,] : L \times L \to L$$

(called the **bracket** or **commutator**) such that:

1. $[x, x] = 0$ for all x in L;
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all x, y, z in L. (Jacobi identity)

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Some Examples

Example

$\mathbf{R}^3$ with the Lie bracket given by the cross product of vectors

$$[x, y] = x \times y, \text{for all } x, y \in \mathbf{R}^3 .$$

Example

Let $\mathfrak{gl}(n, k)$ be the vector space of all $n \times n$ matrices over k with the Lie bracket defined by

$$[x, y] = xy - yx,$$

where the multiplication on the right is the usual product of matrices.

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Let $\mathfrak{sl}(2, \mathbf{C})$ be the vector space of all 2×2 trace-free matrices over $\mathbf{C}$ with the Lie bracket defined by

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Example

Let $\mathfrak{sl}(n, k) \subseteq \mathfrak{gl}(n, k)$ consist of the matrices of trace 0. $\mathfrak{sl}(n, k)$ is closed under the Lie bracket, and therefore it is a Lie algebra, called the **special linear algebra**.

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Classification of Simple Lie Algebras

- A Lie algebra is **simple** if it has no non-trivial ideals and is not abelian.
- A Lie algebra is **semisimple** if it does not contain any non-zero abelian ideals.
- In particular, a simple Lie algebra is semisimple.
- Conversely, it can be proven that any semisimple Lie algebra is the direct sum of its minimal ideals, which are canonically determined simple Lie algebras.

Classification

Semisimple Lie algebras over an algebraically closed field have been completely classified.

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The Group Algebra

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Let G be a group and k a field. The **group algebra** $k[G]$ is the set of all linear combinations of finitely many elements of G with coefficients in k.

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The group algebra is a Lie algebra.

Structure Theorem

Let $\mathfrak{L}(G)$ be the subspace of $\mathbf{C}[G]$ that is the linear span of the elements $\hat{g} = g - g^{-1}$. Then $\mathfrak{L}(G)$ is a Lie-subalgebra of $\mathbf{C}[G]$.

What Lie algebra is it?

Theorem

The Lie algebra $\mathfrak{L}(G)$ admits the decomposition

$$\mathfrak{L}(G) = \bigoplus_{\chi \in \mathfrak{R}} \mathfrak{o}(\chi(1)) \oplus \bigoplus_{\chi \in \mathfrak{Sp}} \mathfrak{sp}(\chi(1)) \oplus \bigoplus_{\chi \in \mathfrak{C}}{}' \mathfrak{gl}(\chi(1))$$

where $\mathfrak{R}, \mathfrak{Sp}$ and $\mathfrak{C}$ are the sets of irreducible characters of real, symplectic, and complex types, respectively, and where the prime signifies that there is just one summand $\mathfrak{gl}(\chi(1))$ for each pair $\{\chi, \bar{\chi}\}$ from $\mathfrak{C}$.

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Example

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Consider the group S_3. Since

$$(1,2) = (1,2)^{-1} \tag{1}$$
$$(1,3) = (1,3)^{-1} \tag{2}$$
$$(2,3) = (2,3)^{-1}, \tag{3}$$

we have

$$\widehat{(1,2)} = \widehat{(1,3)} = \widehat{(2,3)} = \hat{0}.$$

Also, $\widehat{(1,2,3)} = (1,2,3) - (1,3,2)$, so

$$\mathfrak{L}(S_3) = \{\hat{0}, \widehat{(1,2,3)}\} = \text{span}\{\widehat{(1,2,3)}\}.$$

Thus $\dim \mathfrak{L}(S_3) = 1$.

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Example continued

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The group S_3 has 3 characters, all of real type, of degrees 1, 1, 2. So, by the above theorem $\mathfrak{L}(S_3)$ decomposes in the following way:

$$\mathfrak{L}(S_3) = \mathfrak{o}(1) \oplus \mathfrak{o}(1) \oplus \mathfrak{o}(2).$$

By adding up the dimensions of the irreducibles in the decomposition, we get

$$0 + 0 + 1 = 1.$$

My project

$\mathfrak{L}(G)$ is a Lie-subalgebra of $k[G]$ for any field k.

Question

Can we find a similar structure theorem if we take k to be a finite field instead of $\mathbf{C}$?

- Classification of Lie algebras over finite fields is MUCH more complicated.
- Representations of groups over finite fields is also much more complex than over an algebraically closed field.

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