Lie Groups

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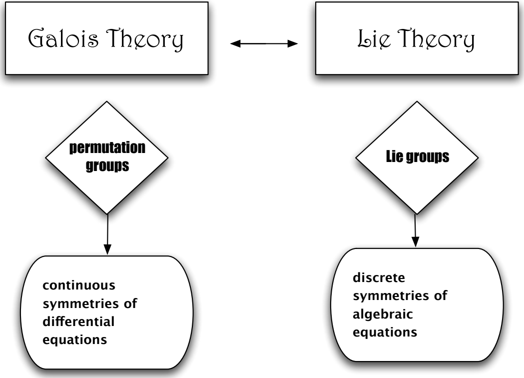
Definition

Intuitively, a Lie group is a group which is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure.



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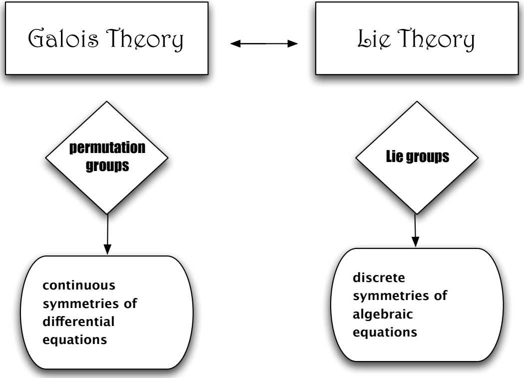

Motivation

Consider the set of all rotations in 3-space (leaving the origin fixed). These are denoted **SO(3)**.

Consider the solid ball in $\mathbb{R}^3$ of radius r. Identify antipodal points.

The ball with antipodal surface points identified is a smooth manifold, and this manifold is diffeomorphic to the set of rotations. So,

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Rotations as Group

By definition, a rotation about the origin is a linear transformation that preserves length of vectors and preserves orientation.

- There is a closed composition law.
- There is an identity element (the identity map).
- Every rotation has a unique inverse rotation.
- The composition law is associative.

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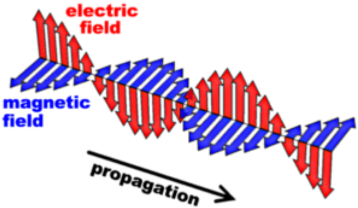

Suppose you want to solve problems which are linear and have spherical symmetry, problems as diverse as compression waves emanating from a central source, electromagetic waves emitted from a small source, or the electron field around an atom.

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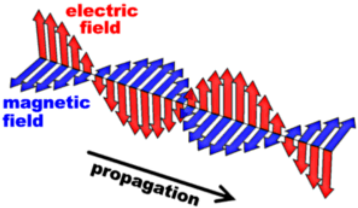

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Formal Definition

Definition

A Lie group is a group G which is also a manifold with a C^∞ structure such that the maps

$$\begin{aligned} (x, y) &\rightarrow xy && \text{from } G \times G \text{ to } G \\ x &\rightarrow x^{-1} && \text{from } G \text{ to } G \end{aligned}$$

are C^∞ functions.

It clearly suffices to require that the map $(x, y) \rightarrow xy^{-1}$ is C^∞.

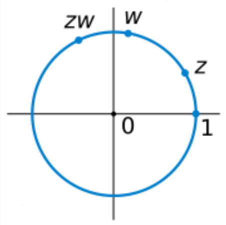
Examples

Example

Euclidean space $\mathbb{R}^n$ is an abelian Lie group (with ordinary vector addition as the group operation).

Example

The circle of center 0 and radius 1 in the complex plane is a Lie group with complex multiplication.



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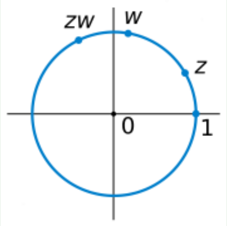

S^1 as $\mathbb{R}/\mathbb{Z}$

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi it} : t \in \mathbb{R}\} = \{e^{2\pi it} : 0 \leq t < 1\}.$$

Let ϕ be the map $t \to e^{2\pi it}$ from $\mathbb{R}$ onto the unit circle, and note that ϕ is a homomorphism:

$$\phi(s+t) = e^{2\pi i(s+t)} = e^{2\pi is}e^{2\pi it} = \phi(s)\phi(t).$$

Furthermore, the kernel of ϕ is exactly $\mathbb{Z}$, so indeed, we have a group homomorphism from S^1 to the quotient group $\mathbb{R}/\mathbb{Z}$.

Other Important Examples

Example

The group $GL_n(\mathbb{R})$ of invertible matrices (under matrix multiplication) is a Lie group of dimension n^2, called the general linear group. It has a subgroup $SL_n(\mathbb{R})$ of matrices of determinant 1 which is also a Lie group, called the special linear group.

Example

The orthogonal group $O_n(\mathbb{R})$ is a Lie group represented by orthogonal matrices. It consists of all rotations and reflections of an n-dimensional vector space. It has a subgroup $SO_n(\mathbb{R})$ of elements of determinant 1, called the special orthogonal group or rotation group.

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Lie's Idea

One of the key ideas in the theory of Lie groups, due to Sophus Lie, is to replace the global object, the group, with its local or linearised version, which Lie himself called its “infinitesimal group" and which has since become known as its Lie algebra.

To every Lie group, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group.

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Lie Algebra

Definition

A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called *the bracket*) such that for all $x, y, z \in \mathfrak{g}$,

$$(a) \qquad [x, y] = -[y, x]. \qquad \text{(anti-commutativity)}$$

$$(b) \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \qquad \text{(Jacobi identity)}$$

How to get the Lie Algebra of a Lie Group

- Vector fields on a smooth manifold (derivations of the ring of smooth functions on the manifold) form a Lie algebra under the Lie bracket $[X, Y] = XY - YX$.
- Any group G that acts smoothly on the manifold acts on the vector fields; the vector space of vector fields fixed by the group is closed under the Lie bracket and therefore also forms a Lie algebra.

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How to get the Lie Algebra of a Lie Group (continued)

- Since a Lie group is a manifold, we let it act as a group on the manifold by left translations and obtain the Lie algebra formed by the left invariant vector fields under the Lie bracket of vector fields.
- Any tangent vector at the identity of a Lie group can be extended to a left invariant vector field by left translating the tangent vector to other points of the manifold. This identifies the tangent space T_eG at the identity with the space of left invariant vector fields, and therefore makes the tangent space into a Lie algebra, called the Lie algebra of G.

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Important Theorems

Theorem

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and G simply connected. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = \phi$.

Corollary

If simply connected Lie groups have isomorphic Lie algebras, then they are isomorphic.

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As a consequence of this, if $\mathfrak{g}$ is a Lie algebra, then there is a Lie group, in particular a simply connected one, with Lie algebra $\mathfrak{g}$. In view of this we have:

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