Toward an Online Knowledgebase for Knots and Quandles

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Abstract

The goal of my research project at Bard has been to devise a way to share ASC Lab research work over the World Wide Web. The project consisted of two main parts. The first involved verifying knot colorings by the Wood6 and Sharac-n quandles and creating the images for posting. The second part involved the development of an ontology for finite quandles in order to support semantically rich content. We hope that this ontology will eventually provide an efficient way for anybody studying different algebras to search for and share information.

Contents

1 Introduction

The ASC Lab has acquired valuable results on the subject of knots and quandles. We have a very large set of images, depicted knots, colored by specific quandles, some of which have been colored by quandles generated in our lab (i.e. Merling, Golbus, Sharac quandles). In order to share this information, I have been working on an ontology. As it stands, researchers have to locate and contact each other to exchange information. When a researcher determines something new about an algebra, the information is either buried or it is posted on the web where it might not easily be accessible. Creating a knowledgebase of information is very important to the advancement of research.

2 Knot Images

The goal of this work has been to complete a set of images of colored knots. In order to create knot images, the knots had to be colored by a quandle. The specific quandles I used to color knots include Wood6 and Sharac-n.

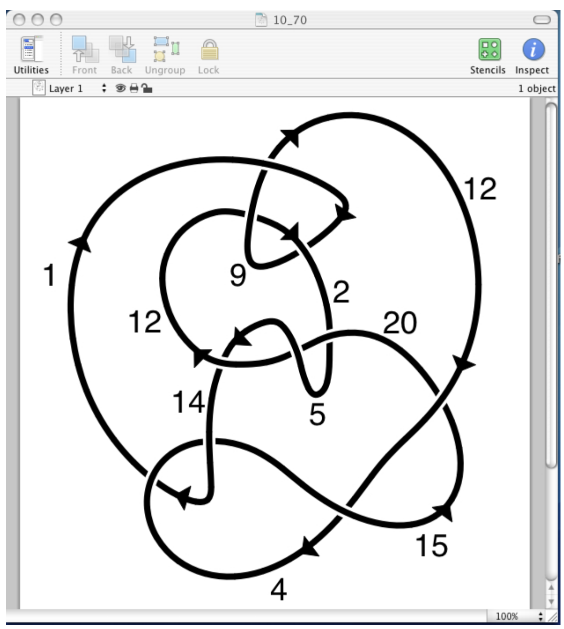
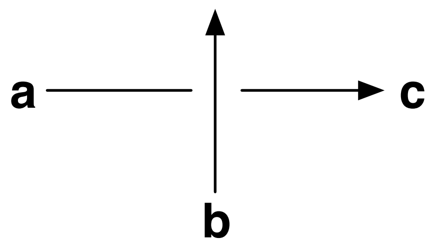


Figure 1: A finished image of knot 10_70 colored by Sharac6 in Omnigraffle

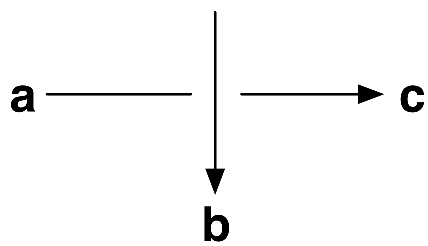
Definition: A **quandle** $\langle Q, \triangleright \rangle$ is a set Q along with a binary operation $\triangleright$ that satisfies the following conditions:

- *Idempotence*: $x \triangleright x = x$.
- *Right cancellation*: If $x \triangleright r = y \triangleright r$ then $x = y$.
- *Right self-distribution*: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Definition: A **coloring** of a knot $\mathcal{K}$ by a quandle Q is a labeling of the arcs of a knot $\mathcal{K}$ by the elements of Q that uses at least two elements of Q and satisfies the following conditions at each crossing.



$a \triangleright b = c$



$a = c \triangleright b$

After other members of the ASC Lab team colored and verified all of the knot colorings, I would take the Mathematica[33] image print-outs and find each knot in KnotPlot[30], a program that contains a variety of knot images easy to manipulate. Next I would rotate each knot in Knotplot to follow the orientation of the mathematica images, which would include flipping the knot across the x and y axis, and using a feature that allows you to move the strings of the knot and untwist extraneous loops. I would set a specific width for the knot, and save it as an eps file. Each knot would be opened in Omnigraffle[13], a diagramming application, and the colorings would be labeled. The last step in creating the knot images was placing arrows facing the correct orientation on the image. I essentially created a library of black and white knot images with the proper colorings. The next step in this project would be a prototype knot coloring digital library. The ASC Lab is consistently discovering new quandles which can color knots and this project will most likely be passed onto an ASC Lab newcomer.

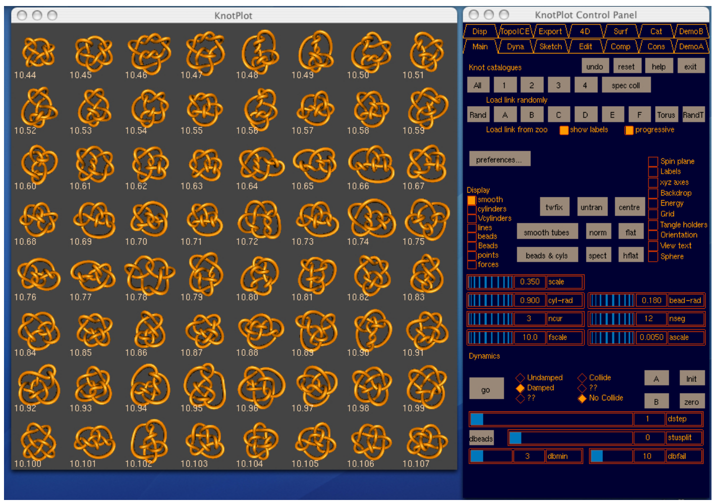


Figure 2: KnotPlot's collection of knot images

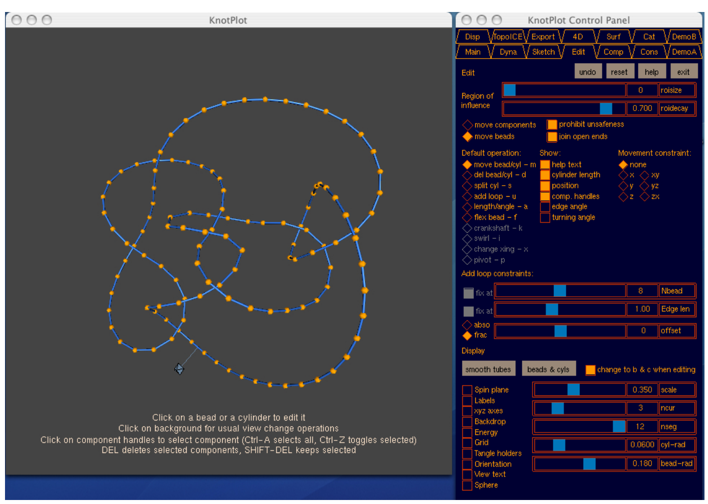


Figure 3: KnotPlot's ability to rotate and twist knot sections

3 Quandle Ontology

The main problems with the way information is shared in the algebraic world deal with interoperability and web search. Researchers studying general universal algebras of a computational nature will use programs such as KnotPlot[30], Prolog[32], Mathematica/Maple/Sage[33], Mace/Prover9[19], UACalc[11] etc... for their work. Because these programs do not have a common format, they do not allow what one has created to catch the eye of search engines. The solution to this problem is to create a common computational vocabulary. Programs can be designed to search for specific words and recognize XML[4]. Searches would become more efficient. An ontology is a means to build this kind of vocabulary. Its format gives it a semantic advantage to help search engines pick up key words. It would revolutionize the exchange of ideas in universal algebra and allow people to share their research information in an organized way.

3.1 Quandle Ontology

As part of the solution to creating a space where people can share information, we are specifically creating a quandle ontology. The quandle ontology is a roadmap of vocabulary for all of the research on the subject of finite quandles[15] thus far. It contains a hierarchy of different types of quandles, as well as their relationships with each other and key words such as other names these quandles have been given. We hope that this ontology will eventually provide an efficient way for anybody studying different algebras to search for and share information. For instance, it took our research team several days to run a process that built a quandle of size 28 in Mace. With a semantically intelligent search, a user looking for this type of quandle would be able to access it on the web. An ontology would help facilitate the flow of ideas in algebraic research and save users from having to perform a long search.

3.2 Design of an Ontology

The bulk of the design process has been drafting the relationships between classes and searching researching papers for quandle vocabulary. Protege[12] is an ontology editor which uses OWL[20] to build the hierarchy of classes. It allows for the addition of classes and subclasses, restrictions and conditions, comments, disjoints, properties, individuals and comments. Many questions have surfaced throughout the process of creating an ontology for quandle vocabulary. Some were resolved quickly, others needed more consideration and have yet to be resolved. Decisions had to be made about the definitional vs. consequential issues in designing an ontology. When it came to the question of when to subclass, we decided that it would be better to keep classes which we did not know everything about on the same level. We would only subclass when we were positive of the relationship– for example, we know that the conjugation quandle is a subclass of the n-fold conjugation quandle because it contains all of the properties of its superclass plus something more, the former is a subclass of the latter by definition. By only subclassing relationships we are sure of, when we learn new information in the future that changes the relationships, we can add these facts like a new layer rather then reconstructing the entire ontology. Another issue we are still considering is the technicality of the vocabulary, as well as the conditions and attributes for the classes. We are considering representing words like "finite" as conditions in description logic.

3.3 Additional Questions that have Surfaced

- What levels does the hierarchy of classes contain?
- What properties do we use to link class relationships?
- Will we have attributes?
- We have deleted a Database Tables class which contains the subclasses:
  -Knot Homomorphic Image
  -Homomorphic Image
  -Subquandles
  What will we do with this information?
- Where would cycle structures be placed in our vocabulary?
- Will we be using description logic?

3.4 Facts the Ontology Currently Contains

- All Latin quandles are totally connected.[2] (But not all connected quandles are Latin)
- If Q is not totally connected, then Q is NP-Complete.[22] (But if Q is totally connected, we do not know if it is NP-Complete or not)
- All latin quandles must be tractable.
- Transposition T_n is connected for all n and NP-Complete for n>=4.

- A cyclic quandle is a latin quandle for odd n.[2]
- The extended tait quandle is a latin quandle for odd n.[9]
- Knot quandles are connected.[14]
- Symplectic quandles are "almost connected."[24]
- Latin quandle is part of Merling.
- All Golbus quandles are Latin quandles, and thus must be tractable as well.
- Involutory is also called Kei.
- Unary is also called Trivial.
- Knot Quandle is also called Fundamental Quandle.

3.5 ISWC

My adviser Robert W. McGrail and I submitted a poster/paper titled *Toward an Ontology for Finite Algebras* to the 2008 International Semantic Web Conference in October of 2008. Our paper was accepted and we present our results (cite their page) late October.

3.6 Future Expectations for the Ontology

This ontology will serve as a way to support the static posting of quandles on the web. People researching other algebras would be able to create their own ontologies and share specific rules they have acquired through their research. This project upon completion will be a distributed knowledgebase of algebras. It would allow users to be able to access information quickly and discover commonalities between algebras. We will have a centralized search engine at the ASC Lab with storage here and there will be a way to dynamically generate dihedral and transposition quandles.

3.7 Ontology Images

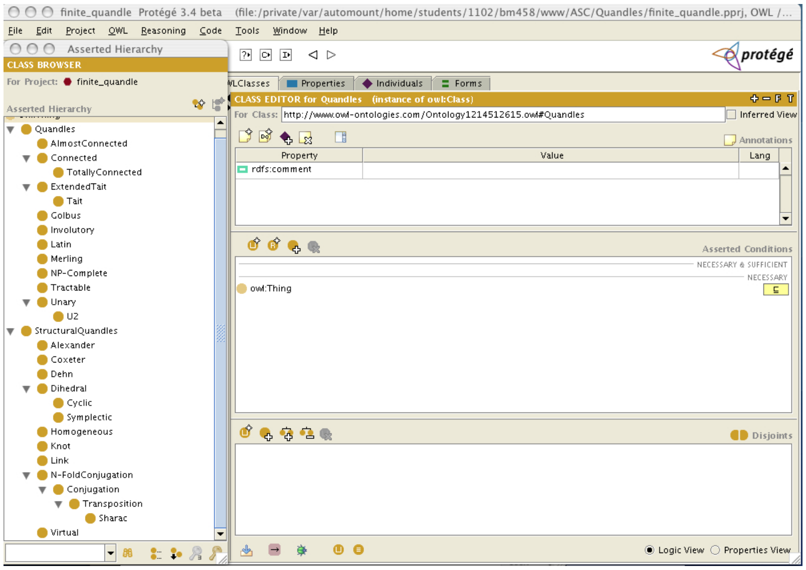


Figure 4: Hierarchy of Classes in Protege

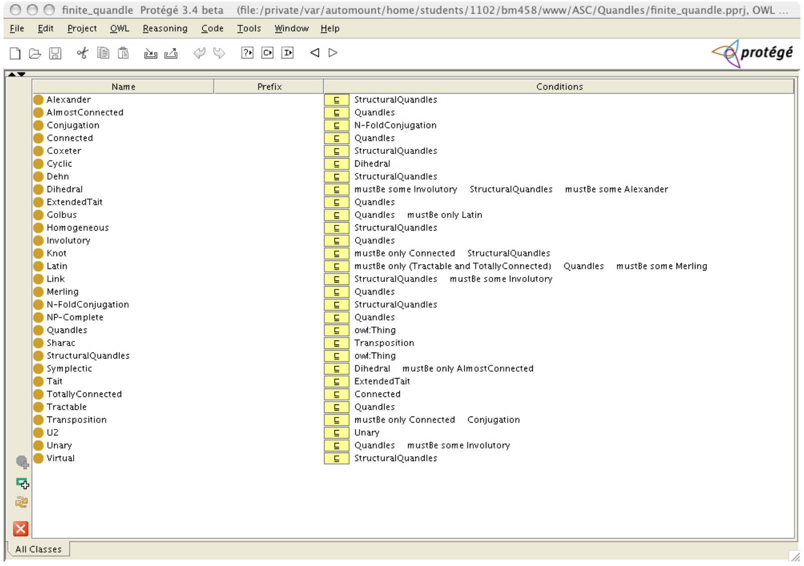


Figure 5: Class Relationships

4 New Technology Skills

When I started research this summer, I had some general ideas of what I was going to do, but there was a lot of trial and error in the beginning. I learned a little about different computer languages, such as Links, HTML/XHTML, Prolog and LaTeX. Although I did not end up using these languages in the creation of my two main projects, the ontology and knot coloring, nothing went wasted. I now have a greater understanding of a larger variety of topics in computer science.

4.1 Links

Links was the first language I looked at. Most web programs require a knowledge of several languages. The logic, presentation, GUI behavior and queries are written differently from each other. Since it is difficult to link these different kinds of code, an impedance mismatch problem tends to occur. Links is a single language that handles all three tiers, the browser, server, and database. It is a relatively new language and does not have a lot of documentation.

4.2 HTML/XHTML

I read the text *HTML & XHTML: The Definitive Guide, 6th Edition* from an O'Reilly website subscription. I tried out sample code in Aquamacs and was able to use it with a CSS style sheet to see the result in a web browser. These languages are predominately used to create web pages.

4.3 Prolog

Prolog[32] is a logic based programming language and a very useful tool in quandles research. I got to experience learning how to use it through books and tutorials, and I was able to practice and write minimal code. I tried to use Prolog to create a little program that converts one specific atom in a list to three different atoms. I also converted facebook markup language (FBML) tags into HTML within Prolog.

4.4 LaTeX

LaTeX is the document markup language I got familiar with in order to create this document, a research poster presentation of my work, and a document submitted to the 2008 International Semantic Web Conference.

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