Toward Quandle Dichotomy

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Quandles

Definition

A **quandle Q** $= \langle Q, \triangleright \rangle$ is a set Q along with a binary operation $\triangleright$ that satisfies the following conditions:

- (Idempotence:) $x \triangleright x = x$.
- (Right Cancellation:) If $x \triangleright r = y \triangleright r$ then $x = y$.
- (Right Self-Distributivity:) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Define *homomorphism* of quandles and *subquandles* in the usual way.

Example

$\triangleright$	0	1	2
0	0	0	1
1	1	1	0
2	2	2	2

Table: Quandle of size 3

Motivation: Group Conjugation

Proposition

Let G be a group. Define the operation $\triangleright$ on G as follows. For $a, b \in G$ let

$$a \triangleright b = b^{-1}ab$$

Then $(G, \triangleright)$ is a quandle.

Proof.

- $a \triangleright a = a^{-1}aa = (a^{-1}a)a = ea = a.$
- $\ldots$

$\square$

$(G, \triangleright)$ is a **group quandle**. If $Q \subseteq G$ is closed under $\triangleright$ then $(Q, \triangleright)$ is a **conjugation quandle**.

Unary Example

$\triangleright$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

Table: Unary Quandle

Note: This is the conjugation quandle of $\mathbb{Z}_3$.

Quasigroup Example

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Table: Quasigroup Quandle

Note: This is a subquandle of $(S_3, \triangleright)$.

Quasigroup Example

$\triangleright$	0	1	2
0	0	2	1
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2	1	0	2

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Note: This is a subquandle of $(S_3, \triangleright)$.

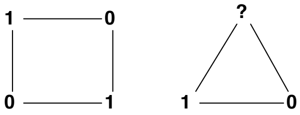
Definition

A **quasigroup quandle** is a quandle that satisfies left-cancellation, so if $r \triangleright x = r \triangleright y$ then $x = y$.

2-Color

Definition

A graph $G = (V, E)$ is *2-colorable* if one can assign to each vertex $v \in V$ one of two colors, say 0 and 1, in such a way that for each edge $(v, v') \in E$, v and v' have been assigned different colors.



Note: 2-Color is tractable.

2-Color as CSP

Proposition

2-Color *is a constraint satisfaction problem.*

Let $\{0, 1\}=$ values, $V=$ variables. For every $(v, v') \in E$, include the constraint

$$\langle (v, v') \in \{(0, 1), (1, 0)\} \rangle.$$

Other CSP's: Sat, n-Queens, Schedule, Sudoku

Inv(Q)

Definition

Let Q be a finite quandle and $R \subseteq Q^n$ be a relation over the set Q of arity n. Then R is **invariant under Q** if

$$(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in R$$

implies

$$(a_1 \triangleright b_1, a_2 \triangleright b_2, \ldots, a_n \triangleright b_n) \in R.$$

Let **Inv(Q)** stand for the set of all relations invariant under Q.

CSP over Q

Definition

Let Q be a quandle. Any CSP over the set Q which employs only relations from $Inv(Q)$ is called a **CSP over Q**.

Definition

A quandle Q is **tractable** if every CSP over Q is solvable in polynomial time. Q is **NP-complete** if at least one CSP over Q is NP-complete.

U_2

U_2 is the only quandle of size 2.

$\triangleright$	0	1
0	0	0
1	1	1

Table: U_2

- $Inv(U_2)$ includes all relations over $\{0, 1\}$.
- 2-Color and 3-SAT are CSP's over U_2.
- U_2 is NP-complete.

Factors and Reductions

Definition

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Corollary

Q is NP-complete if it has an NP-complete factor.

Inner Automorphism Group

For $q \in Q$ define $\sigma_q : Q \to Q$ by

$$\sigma_q(x) = x \triangleright q.$$

Then σ_q is an monomorphism of Q. If Q is finite, σ_q is an automorphism.

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Definition

Let $Inn(Q)$ be the subgroup of Sym_Q generated by $\{\sigma_q | q \in Q\}$. We will call it the *inner automorphism group of Q* .

Connected and Totally Connected Quandles

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Definition

A quandle is **totally connected** if all of its subalgebras are *connected* .

Example of a disconnected quandle: Sharac 5

$\triangleright$	0	1	2	3	4
0	0	0	0	0	1
1	1	1	1	1	0
2	2	2	2	2	3
3	3	3	3	3	2
4	4	4	4	4	4

Table: Sharac 5

The orbits are $\{0,1\}$, $\{2,3\}$, $\{4\}$.

Example: Transposition Quandle

All quasigroup quandles are (totally) connected. However, there are connected quandles which are not quasigroups. For example:

$\triangleright$	0	1	2	3	4	5
0	0	3	4	1	2	0
1	3	1	5	0	1	2
2	4	5	2	2	0	1
3	1	0	3	3	5	4
4	2	4	0	5	4	3
5	5	2	1	4	3	5

Table: T_4

Disconnected Quandles

Theorem

If Q is not connected, then Q is NP-complete.

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Proof.

For $q \in Q$, define the homomorphism $H : Q \to U_2$ as follows.

$$H(q) = \begin{cases} 0, & q \in (x\text{-orbit}) \\ 1, & \textit{otherwise} \end{cases}$$

If Q is not connected, then h is surjective. □

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Corollary

If Q is not totally connected, then Q is NP-complete.

Quandle Dichotomy Conjecture

Conjecture (Quandle Dichotomy)

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Every CSP is tractable or NP-complete.

This is only moderately daring since failure implies $\mathrm{P} \neq \mathrm{NP}$!

The Current State of Quandle Dichotomy