Tricolorable Torus Knots are NP-Complete

Peter Golbus*, Robert W. McGrail, Tomasz Przytycki, Mary Sharac and Aleksandar Chakarov
The Laboratory for Algebraic and Symbolic Computation
Reem-Kayden Center for Science and Computation
Bard College
Annandale-on-Hudson, NY 12504
{pg237, mcgrail, tp884, ms182, ac354}@bard.edu

ABSTRACT

This work presents a method for associating a class of constraint satisfaction problems to a three-dimensional knot. Given a knot, one can build a knot quandle, which is generally an infinite free algebra. The desired collection of problems is derived from the set of invariant relations over the knot quandle, applying theory that relates finite algebras to constraint satisfaction problems. This allows us to develop notions of tractable and NP-complete quandles and knots. In particular, we show that all tricolorable torus knots and all but at most 2 non-trivial knots with 10 or fewer crossings are NP-complete.

1. INTRODUCTION

Since Cook's formulation of NP-completeness [4], computer scientists have labored to unravel the mysteries of nondeterministic polynomial time [27]. Early efforts included the building of a catalogue of individual NP-complete combinatorial problems in the hope that one or more would provide significant insight [17]. In the meantime, more structurally oriented approaches have emerged that instead focus on subclasses of NP. A notable example is the development of descriptive complexity [6, 12], which considers complexity classes axiomatized by fragments of (existential) second-order logic.

Another promising avenue restricts attention to subclasses of CSP, the class of constraint satisfaction problems [21]. Early on, Schaefer proved that every Boolean constraint satisfaction problem is NP-complete or tractable [25]. Feder and Vardi conjectured that this dichotomy holds for all of CSP [7]. Since then, Bulatov has extended Schaefer's result to three-element domains [2].

More importantly, Feder and Vardi showed that a solution to a constraint satisfaction problem corresponds to a homomorphism between certain finite, first-order structures. This idea was further refined by Jeavons and others [13, 14], and has led to significant insight into the structure of tractable subclasses of CSP [9, 10]. In particular, Jeavons, Cohen, and Pearson explored the relationship between CSP and universal algebra [15].

In [3], Bulatov, Jeavons, and Krokhin used the language of relational clones [28] and tame congruence theory [11, 20] to formulate notions of tractable and NP-complete algebras. They showed that in order to classify finite algebras as tractable or NP-complete, one need only consider the surjective algebras. They also proved P/NP-complete dichotomy for finite strictly simple surjective algebras. Moreover, they identified the class of idempotent algebras, all of which are surjective, as a prime target for the next round of dichotomy results.

A compelling example class of idempotent algebras is the variety [11, 20] of quandles [16]. Quandles were originally inspired by knot theory; a significant class of quandles are related to 3-dimensional knots. We use this relationship to define a notion of constraint satisfaction problem over a knot [8]. In this context we are able to show that all tricolorable torus knots and all but at most 2 non-trivial knots with 10 or fewer crossings are NP-complete.

2. KNOTS AND QUANDLES

2.1 Knot Basics

The basics of knot theory are reviewed in this section; more extensive treatments can be found in [5, 24]. A **knot** $\mathcal{K}$ is a smooth embedding of the unit circle S^1 into $\mathbf{R}^3$. A knot $\mathcal{K}$ is usually identified with its oriented image in $\mathbf{R}^3$. Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are **ambient isotopic** if $\mathcal{K}_1$ can be continuously deformed into $\mathcal{K}_2$ and vice versa.

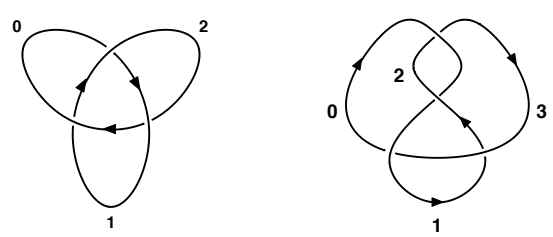

Figure 1: Trefoil (3_1) and Figure Eight (4_1) Knots

Knots are often represented by their 2-dimensional projections, as the Trefoil and Figure Eight knots are in Figure 1. Notice that the projection of the Trefoil has three crossings. Any knot that is ambient isotopic to the Trefoil will have at least three crossings in all of its projections. A knot projection that realizes the minimum possible number of crossings

ACMSE 2009 March 19-21, 2009, Clemson, South Carolina, USA

is called **reduced**.

Each crossing of a knot projection causes an apparent break in the segment of the strand below the crossing. For the duration of this article, the unbroken segments of the strand are called **arcs**. Each arc is labeled by an integer.

Many knots can be specified using **Alexander-Briggs** notation. For example, the Trefoil knot of Figure 1 is denoted 3_1. In general, the Alexander-Briggs specification of a knot is n_k where n is the number of crossings and k is nominal.

A **link** is a collection of knots that do not intersect but may be entangled. Alternatively, a knot can be defined as a link with exactly one component.

2.2 Quandles

Quandles are an algebraic invariant of knots developed by Joyce [16].

Definition 1. A quandle $\mathbf{Q} = (Q, \{\triangleright, \blacktriangleright\})$ is a set Q together with binary operations $\triangleright, \blacktriangleright$: $Q \times Q \to Q$ satisfying the following axioms:

Idempotence: $\forall x(x \triangleright x = x)$;
Right Cancellation A: $\forall xy((x \triangleright y) \blacktriangleright y = x)$;
Right Cancellation B: $\forall xy((x \blacktriangleright y) \triangleright y = x)$; and
Right Self-Distributivity:

$$\forall xyz((x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)).$$

The simplest examples of quandles are the **unary quandles $\mathbf{U_n}$** where n is a positive integer. The underlying set of $\mathbf{U_n}$ is $\{0, 1, \ldots, n-1\}$ and the operations $\triangleright$ and $\blacktriangleright$ simply project the first argument:

$$x \triangleright y = x \blacktriangleright y = x.$$

$\triangleright, \blacktriangleright$	0	1
0	0	0
1	1	1

$\triangleright, \blacktriangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Figure 2: The Cayley Tables for $\mathbf{U_2}$ and $\mathbf{D_3}$

The **dihedral quandle $\mathbf{D_n}$** has the same underlying set as $\mathbf{U_n}$, but its operations are defined by

$$x \triangleright y = x \blacktriangleright y = 2y - x (\text{mod } n).$$

Figure 2 displays the operation tables for $\mathbf{U_2}$ and $\mathbf{D_3}$.

2.3 The Knot Quandle

Given a projection for a knot $\mathcal{K}$, one can construct a **quandle presentation**, $\mathcal{Q}(\mathcal{K})$, as follows: To each crossing, assign a simple identity using the relevant arc labels and one of the two binary operations, $\triangleright$ or $\blacktriangleright$, depending on the orientation of the arcs as in Figure 3.

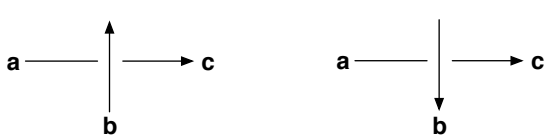

Figure 3: Left and Right Crossings

In the left diagram, arc a passes under arc b, becoming arc c, with arc b's orientation to the left of arc c's. Therefore this translates to the equation

$$c = a \triangleright b.$$

The right diagram of Figure 3 has b crossing to the right instead, which corresponds to

$$c = a \blacktriangleright b.$$

For example, the Trefoil projection of Figure 1 has the following presentation:

$$\mathcal{Q}(3_1) = \langle 0, 1, 2 | 1 = 0 \triangleright 2, 2 = 1 \triangleright 0, 0 = 2 \triangleright 1 \rangle.$$

2.4 The Reidemeister Moves

That quandles are an algebraic invariant of knots can be shown by examining the **Reidemeister moves**. Reidemeister [23] proved that two knots are ambient isotopic if and only if one can be deformed into the other through successive applications of three types of transformations, called Reidemeister moves, and planar deformations. To apply a move, we focus on some small region of the knot projection. If that part resembles one of the two diagrams of the move, it may be transformed to resemble the other diagram. It is assumed that the rest of the knot remains unchanged during this deformation. This results in a knot that is ambient isotopic to the previous one.

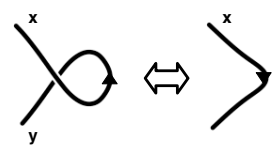

Figure 4: Type I Reidemeister Move

An example of a **Type I** move appears in Figure 4. The left hand diagram has a segment of the knot looping behind itself. The crossing forms two arcs, x and y, and thus corresponds to the equation

$$y = x \triangleright x.$$

A simple twist of the loop yields the right hand diagram, reducing this part to one arc x. Here the role of y within the rest of the knot is now fulfilled by x. Hence

$$x = y = x \triangleright x,$$

so from ambient isotopy, one may infer that $\triangleright$ is idempotent.

Figure 5 presents an instance of a **Type II** move. The

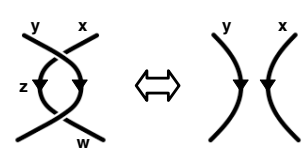

Figure 5: Type II Reidemeister Move

diagram on the left has arc y crossing over two points of the knot in succession, while on the right, y has been placed so that these two crossings do not occur. The point on the

left hand diagram labeled by w is equated with its analogous location in the other diagram. Thus, we have Right Cancellation A.

$$x = w = z \blacktriangleright y = (x \triangleright y) \blacktriangleright y.$$

Reversing the orientation on arc y leads to the Right Cancellation B identity.

Lastly, right self distributivity is derived from a **Type III** move (Figure 6). In this scenario, there are two segments

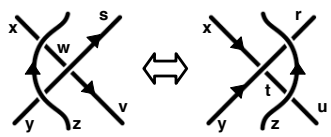

Figure 6: Type III Reidemeister Move

that form one crossing in the center of both diagrams, and a third, single-arc segment z that crosses over the other two segments. The diagrams differ as to whether z crosses to the left or right of the central crossing. From this move, one can infer that

$$v = u.$$

Analyses of the crossings in both diagrams yield

$$(x \triangleright y) \triangleright z = t \triangleright z = u = v = w \triangleright s = (x \triangleright z) \triangleright (y \triangleright z).$$

The quandle axioms guarantee that ambient isotopic knots, as well as different projections of the same knot, have isomorphic knot quandles. Hence, the quandle presentation $\mathcal{Q}(\mathcal{K})$ of Section 2.3 is well defined.

Since the right cancellation identities ensure that the equation

$$x \triangleright y = z$$

is provably equivalent to

$$x = z \blacktriangleright y,$$

the operation $\blacktriangleright$ is uniquely determined by $\triangleright$. One may dispense entirely with $\blacktriangleright$. Therefore, to show that $\mathbf{Q}'$ is a subquandle of a quandle $\mathbf{Q}$, it suffices to show that $\mathbf{Q}'$ is closed under $\triangleright$. Similarly, a function $h\colon \mathbf{Q} \to \mathbf{Q}''$ is a quandle homomorphism if it preserves $\triangleright$. Henceforth, finite quandles will be presented via the Cayley table for $\triangleright$ alone. We also often eliminate $\blacktriangleright$ from quandle presentations: $\mathcal{Q}(4_1)$ (Figure 1), which has both types of crossings, can be expressed as

$$\mathcal{Q}(4_1) = \langle 0,1,2,3 | 0 = 1 \triangleright 2, 2 = 1 \triangleright 3, 2 = 3 \triangleright 0, 0 = 3 \triangleright 1 \rangle.$$

2.5 Tricolorable Knots and Finite Images

A precursor to Joyce's concept of quandle is **tricolorability** [22]. A tricoloring of a knot $\mathcal{K}$ is an assignment of one of three colors $\{0,1,2\}$ to each arc of $\mathcal{K}$ in such a way that every crossing either has three arcs of the same color or one arc of each color, and such that at least two distinct colors are employed.

For example, the integer labels in Figure 1 constitute a tricoloring of 3_1. One may also view these labels as elements of $\mathbf{D_3}$ (Figure 2). In fact, the equations of $\mathcal{Q}(3_1)$ hold in $\mathbf{D_3}$. In general, a tricoloring of $\mathcal{K}$ corresponds to a quandle homomorphism $h\colon \mathcal{Q}(\mathcal{K}) \to \mathbf{D_3}$.

Important to the development of Section 3.3 are situations in which there is a knot $\mathcal{K}$, a finite quandle $\mathbf{Q}$, and a quandle homomorphism $h\colon \mathcal{Q}(\mathcal{K}) \to \mathbf{Q}$. Such a homomorphism will be called a **$\mathbf{Q}$-coloring** of $\mathcal{K}$. This corresponds to a labeling of the arcs of $\mathcal{K}$ with at least two of the elements of $\mathbf{Q}$ such that at each crossing, the equations of $\mathbf{Q}$ hold. If the homomorphism is surjective, we call $\mathbf{Q}$ a **$\mathcal{K}$-quandle**. For example, it was shown above that $\mathbf{D_3}$ is a 3_1-quandle.

3. CONSTRAINT SATISFACTION PROBLEMS OVER KNOTS

3.1 Constraint Satisfaction Problems over Finite Quandles

We are interested in $\mathcal{K}$-quandles because it allows us to associate finite algebras to a knot. Building on the work of Jeavons and others [15], we can use these finite algebras to construct a notion of constraint satisfaction problem over a knot via the following definitions:

Definition 2. An n-ary relation $R \in Q^n$ is **invariant under** $\triangleright$ if

$$(a_1 \triangleright b_1, \ldots, a_n \triangleright b_n) \in R,$$

whenever $\mathbf{a}, \mathbf{b} \in R$. The set of all finitary relations invariant under $\triangleright$ is denoted $\mathbf{Inv}(\mathbf{Q})$.

Definition 3. Given a finite quandle $\mathbf{Q}$, the constraint satisfaction problem CSP($\mathbf{Q}$) is the combinatorial decision problem with the following components:

Instance: An instance of CSP($\mathbf{Q}$) is a triple,

$$\mathcal{I} = (V', Q, \mathcal{C}),$$

where V' is a subset of a countably infinite set of variables V and $\mathcal{C}$ is a finite set of **constraints over** $\mathbf{Inv}(\mathbf{Q})$. A constraint takes the form of the pair

$$\langle (v_1, \ldots, v_n), R \rangle$$

where R is a set of n-ary relations in $\mathbf{Inv}(\mathbf{Q})$.

Solution: A solution to an instance $\mathcal{I}$ of CSP($\mathbf{Q}$) is a function $\theta\colon V \to Q$ such that for every constraint

$$\langle (v_1, v_2, \ldots, v_n), R \rangle \in \mathcal{C},$$

we have

$$(\theta(v_1), \theta(v_2), \ldots, \theta(v_n)) \in R.$$

Example 1. Consider the Boolean Satisfiability question. That is, does there exist a truth assignment for a proposition such as the following:

$$\alpha = (\neg v_1 \vee v_2) \wedge (v_3 \vee \neg v_2)?$$

Identifying 0 with false and 1 with true, this translates to the following constraint system:

$$C_1 = \langle (v_1, v_2), S_1 \rangle$$

and

$$C_2 = \langle (v_3, v_2), S_2 \rangle,$$

where

$$S_1 = \{(0,0),(0,1),(1,1)\}$$

and

$$S_2 = \{(0,0),(1,0),(1,1)\}.$$

Notice that for all $\mathbf{x},\mathbf{y}$ in the unary quandle $\mathbf{U_2} \times \mathbf{U_2}$ (see Figure 2),

$$(x_1 \rhd y_1, x_2 \rhd y_2) = \mathbf{x}.$$

Thus if $\mathbf{x},\mathbf{y} \in S_i$, then $\mathbf{x} \rhd \mathbf{y} \in S_i$. It follows that both C_1 and C_2 are constraints over $\mathbf{Inv}(\mathbf{U_2})$. Therefore the triple $\mathcal{I} = (\{v_1,v_2,v_3\}, \mathbf{U_2}, \{C_1,C_2\})$ is an instance of CSP($\mathbf{U_2}$).

Example 1 is an instance of 2-SAT, which is a tractable constraint satisfaction problem. Generally, any instance of n-SAT is also an instance of CSP($\mathbf{U_2}$). Since n-SAT is NP-complete for $n > 2$, by Definition 4, $\mathbf{U_2}$ is as well.

Definition 4. A quandle $\mathbf{Q}$ is **tractable** if for every finite $\Gamma' \subseteq \mathbf{Inv}(\mathbf{Q})$, the constraint satisfaction problem CSP(Γ') is tractable. A quandle $\mathbf{Q}$ is **NP-complete** if CSP(Γ') is NP-complete for some finite $\Gamma' \subseteq \mathbf{Inv}(\mathbf{Q})$.

3.2 NP-Complete Quandles

The quandle $\mathbf{U_2}$ plays a central role in this article. As remarked in Example 1, every relation over $\{0,1\}$ is invariant under $\mathbf{U_2}$. Consequently, the set of constraints associated with any NP-complete CSP is a finite subset of $\mathbf{Inv}(\mathbf{U_2})$.

Suppose $\mathbf{Q}'$ is a subquandle of $\mathbf{Q}^n$. Then it follows that $\mathbf{Inv}(\mathbf{Q}') \subseteq \mathbf{Inv}(\mathbf{Q})$. If $\mathbf{Q}'$ is NP-complete, then there exists a finite $\Gamma \subseteq \mathbf{Inv}(\mathbf{Q}')$ such that CSP(Γ) is NP-complete. Clearly, $\Gamma \subseteq \mathbf{Inv}(\mathbf{Q})$ as well, so $\mathbf{Q}$ is NP-complete.

Proposition 1. If $\mathbf{Q}'$ is a subquandle of $\mathbf{Q}^n$ and $\mathbf{Q}'$ is NP-complete, then so is $\mathbf{Q}$.

Idempotence and right cancellation dictate that $\mathbf{U_2}$ is the only quandle of size 2, up to isomorphism. This proves the following:

Corollary 1. Suppose $\mathbf{Q}$ has a subquandle of size 2. Then $\mathbf{Q}$ is NP-complete.

$\rhd$	0	1	2	3	4	5
0	0	3	1	4	2	0
1	2	1	5	0	1	3
2	4	0	2	2	5	1
3	1	5	3	3	0	4
4	3	4	0	5	4	2
5	5	2	4	1	3	5

Figure 7: Sharac$_4$

There appears to be no shortage of NP-complete quandles. Included in this class is the quandle **Sharac$_4$** of Figure 7. Notice that it has $\{0,5\}$ as a subquandle so Corollary 1 applies. Also, **Sharac$_4$** is a 3_1-quandle since it is a homomorphic image of $\mathcal{Q}(3_1)$. This quandle is used in Sections 3.3 and5 to demonstrate the NP-completeness of certain knots.

3.3 Constraint Satisfaction Problems over Knots

Given a knot $\mathcal{K}$, the knot quandle $\mathcal{Q}(\mathcal{K})$ is generally an infinite algebra, and so it does not present an ideal setting for constraint satisfaction problems as formulated in Section 3.1. A more appropriate context is to instead consider a finite, homomorphic image of $\mathcal{Q}(\mathcal{K})$—i.e., a $\mathcal{K}$-quandle.

Definition 5. A **constraint satisfaction problem over** $\mathcal{K}$ is a constraint satisfaction problem over $\mathbf{Q}$ for some $\mathcal{K}$-quandle $\mathbf{Q}$. The knot $\mathcal{K}$ is **tractable** if $\mathbf{Q}$ is tractable for all $\mathcal{K}$-quandles $\mathbf{Q}$, and is **NP-complete** if $\mathbf{Q}$ is NP-complete for at least one $\mathcal{K}$-quandle $\mathbf{Q}$.

4. TRICOLORABLE TORUS KNOTS

4.1 Braids

A **braid** [5] is an intertwining collection of n disconnected strands oriented downward, as in Figure 8. The strands may cross under or over each other but may not turn upward. In other words, the path of each strand in a braid could be traced out by a falling object if acted upon only by gravity and horizontal forces. We consider the leftmost strand to be in position 1. The rightmost strand is in position n.

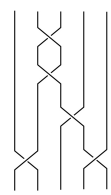

Figure 8: The Braid with Braid Word $\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_4$

All links can be represented as **closed braids** by connecting each strand at the bottom to the strand at the same position at the top. If by following any strand through the braid from top to bottom, you begin and end in the same position, then the closed braid is a link, as depicted on the right in Figure 9. Otherwise it may be a knot, as depicted on the left.

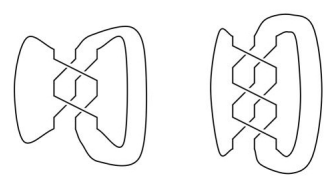

Figure 9: A Knot and a Link Drawn as Closed Braids

A braid can be uniquely identified by a **braid word**. A braid word is constructed by iteratively assigning each crossing a symbol σ_i or σ_i^{-1} where i represents the position of the strand on the left. If the ith strand passes over the $i+1$st strand, we use the symbol σ_i^{-1}. If it passes under, we use σ_i (Figure 10). The braid with braid word $\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_4$ is depicted in Figure 8.

4.2 Torus Knots

A torus link is a link that can be drawn on the surface of a torus without intersection. The torus link $T(p,q)$ is specified

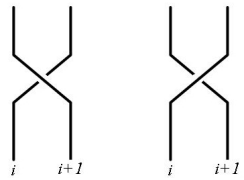

Figure 10: The Crossings σ_i^{-1} and σ_i

by winding p times around the main axis of the torus and q times around the tube of the torus. If the greatest common divisor of p and q is greater than 1, then $T(p,q)$ is a link and not a knot. Furthermore, one can show that $T(p,q)$ is the same link as $T(q,p)$. The simplest torus knot $T(3,2)$ is the Trefoil knot, depicted as a torus knot on the left in Figure 11.

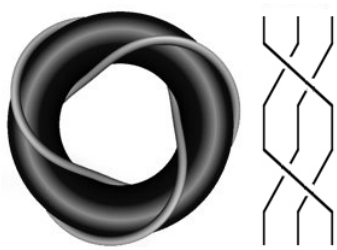

Figure 11: The Trefoil Knot as $T(3,2)$ on a Torus and as a Braid

Torus knots can be drawn very simply as braids via the braid word $(\sigma_1^{-1}\cdots\sigma_{p-1}^{-1})^q$. In Figure 11, the Trefoil is drawn as a braid on the right.

4.3 Quandle Colorings of Braids

Recall from Section 2.5 that a $\mathbf{Q}$-coloring of a knot $\mathcal{K}$ is a quandle homomorphism $h\colon \mathcal{Q}(\mathcal{K}) \to \mathbf{Q}$ that corresponds to a labeling of the arcs of $\mathcal{K}$ with the elements of $\mathbf{Q}$. If $\mathcal{K}$ is $\mathbf{Q}$-colorable, then there will exist an assignment of at least two elements of $\mathbf{Q}$ to the beginning of each strand in a braid representation of $\mathcal{K}$, such that:

1. As the strands travel downward, we will be able to assign new colors at each crossing such that the equations of $\mathbf{Q}$ hold.

2. The color at each position matches at the beginning and the end of the braid.

Given a $\mathbf{Q}$-coloring of a braid, closing it will produce a $\mathbf{Q}$-colored link. It follows from 2. that if an assignment of the elements of $\mathbf{Q}$ produces a $\mathbf{Q}$-coloring of $T(p,q)$, it will also produce a $\mathbf{Q}$-coloring of $T(p,kq)$ for all $k \in \mathbf{N}$. Since $T(p,q)$ is equivalent to $T(q,p)$, it follows that if $T(p,q)$ is $\mathbf{Q}$-colorable then $T(kp,q)$ is also $\mathbf{Q}$-colorable. If an assignment $\mathbf{Q}$-colors $T(p,q)$, it will also $\mathbf{Q}$-color $T(kp,q)$ if we simply repeat our assignment k times. If h is a surjective homomorphism, then the homomorphism that $\mathbf{Q}$-colors $T(kp,q)$ is also surjective. Therfore, if $\mathbf{Q}$ is a $T(p,q)$-quandle, then $\mathbf{Q}$ is also a $T(kp,q)$-quandle.

4.4 Tricolorable Torus Knots are NP-Complete

Recall from Section 3.2, that $\mathbf{Sharac_4}$ is NP-complete. Therefore, if $\mathbf{Sharac_4}$ is a $\mathcal{K}$-quandle of a torus knot $\mathcal{K}$, then $\mathcal{K}$ is NP-complete. To show that all tricolorable torus knots are NP-complete, it will suffice to show that $\mathbf{Sharac_4}$ is a $T(p,q)$-quandle whenever $T(p,q)$ is tricolorable. Since a surjective $\mathbf{Sharac_4}$-coloring of $T(p,q)$ can be extended to generate a surjective $\mathbf{Sharac_4}$-coloring of $T(np,mq)$ for all $n,m \in \mathbf{N}$, it will suffice to show that $\mathbf{Sharac_4}$ is a $\mathcal{K}$-quandle whenever $\mathcal{K}$ is a tricolorable torus knot with p and q both prime. Our search is limited by the following theorem due to Breiland, Oesper and Taalman [1].

Theorem 1. Suppose $T(p,q)$ is a torus knot. Then

1. If p and q are both odd, then $T(p,q)$ is not tricolorable.

2. If p is even and q is odd, then $T(p,q)$ is tricolorable iff $3 \mid q$.

This allows us to prove the following theorem:

Theorem 2. All tricolorable torus knots are NP-complete.

Proof. It follows from Theorem 1 that the only tricolorable torus knot with p and q both prime is $T(2,3)$, which is the Trefoil knot, 3_1. As stated in Section 3.2, $\mathbf{Sharac_4}$ is a 3_1-quandle. □

Since $\mathbf{Sharac_4}$ is a 3_1-quandle, there is a surjective homomorphism whereby $\mathbf{Sharac_4}$-colors 3_1. By the above argument, we can extend this surjective homomorphism to $\mathbf{Sharac_4}$-color any tricolorable torus knot $\mathcal{K}$. Therefore $\mathbf{Sharac_4}$ is a $\mathcal{K}$-quandle, which means that every tricolorable torus knot is NP-complete.

5. ROLFSEN'S KNOT TABLE

The Rolfsen Knot Table [24] includes all knots whose reduced forms have 10 or fewer crossings. For each knot $\mathcal{K}$ in this collection, we determined whether $\mathbf{Sharac_4}$ is a $\mathcal{K}$-quandle. An affirmative answer for $\mathcal{K}$ proved that $\mathcal{K}$ is NP-complete.

Since there exists of a surjective quandle homomorphism $g : \mathbf{Sharac_4} \to \mathbf{D_3}$, if $h : \mathcal{Q}(\mathcal{K}) \to \mathbf{Sharac_4}$ is a surjective homomorphism, then so is $g \circ h : \mathcal{Q}(\mathcal{K}) \to \mathbf{D_3}$. Consequently, we could limit our search for knots with $\mathbf{Sharac_4}$ as a $\mathcal{K}$-quandle to tricolorable knots. Testing all of the tricolorable knots in Rolfsen's Knot Table showed that they were all NP-complete.

To test whether $\mathbf{Sharac_4}$ is a $\mathcal{K}$-quandle for these knots, we developed a program [18][19] written in SWI-Prolog [29] that converted a braid representation of each knot to a quandle presentation and then searched for a nontrivial solution for the presentation in $\mathbf{Sharac_4}$. For good measure, the latter stage of this process was repeated using alternative quandle presentations computed from KnotPlot [26] images. Each positive result was verified by hand computation.

Repeating this process with quandles that are structurally similar to $\mathbf{Sharac_4}$, we were able show that all but at most two of the non-trivial knots in Rolfsen's table are NP-complete.

6. CONCLUSION

The theory developed in Sections 3.1 and 3.3 provide a classifying invariant for knots. While this may prove useful to knot theorists, it was not the purpose of this work.

Rather, the motivation has been to provide a path whereby the tools of knot theory can shed light on a significant subclass of quandles.

Conspicuously absent from this discussion has been any mention of tractable knots. So far, verifying tractability has proved substantially more challenging than demonstrating NP-completeness. At the time of submission, only the Unknot 0_1, which has the trivial knot quandle, is known to be tractable. This remains an active area of research for the ASC lab.

7. ACKNOWLEDGMENTS

The authors would like to acknowledge the contributions of Japheth Wood, a patient explainer of Universal Algebra and a regular source of valuable insight. We would also like to acknowledge Jacqueline Bow, Bella Manoim, Sphurti Saraph, Georgi Smilyanov, Adina-Raluca Stoica, and Petar Stojanov, who verifyied the NP-complete status of many, many knots by hand. Also, Gregory D. Landweber and S. Rebecca Thomas provided many corrections and suggestions toward improving this article.

8. REFERENCES

[1] A.-L. Breiland, L. Oesper, and L. Taalman. p-coloring classes of torus knots. Unpublished manuscript, 2005.

[2] A. A. Bulatov. A dichotomy theorem for constraints on a three-element set. In *43rd IEEE Symposium on Foundations of Computer Science, FOCS '02*, pages 649–658, Vancouver, Canada, November 2002.

[3] A. A. Bulatov, P. Jeavons, and A. A. Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM Journal on Computing*, 34(3):720–742, 2005.

[4] S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd Annual ACM Symposium on the Theory of Computing (STOC'71)*, pages 151–158, Shaker Heights, Ohio, 1971.

[5] P. Cromwell. *Knots and Links*. Cambridge University Press, 2004.

[6] H.-D. Ebbinghaus and J. Flum. *Finite Model Theory*. Monographs in Mathematics. Springer, 2nd edition, 2005.

[7] T. Feder and M. Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM Journal on Computing*, 28:57–104, 1998.

[8] P. Golbus, R. W. McGrail, M. Merling, K. Ober, M. Sharac, and J. Wood. The class of constraint satisfaction problems over a knot. Technical Report number BARD-CMSC-2008-01, Bard College, 2008. http://asc.bard.edu/.

[9] M. Grohe. The structure of tractable constraint satisfaction problems. In *Proceedings of the 31st Symposium on Mathematical Foundations of Computer Science (MFCS'06)*, volume 4162 of *Lecture Notes in Computer Science*, pages 58–72, 2006.

[10] M. Grohe. The complexity of homomorphism and constraint satisfaction problems seen from the other side. *Journal of the ACM*, 54(1), March 2007.

[11] D. Hobby and R. N. McKenzie. *The Structure of Finite Algebras*, volume 76 of *Comtemporary Mathematics*. American Mathematical Society, Providence, R.I., 1988.

[12] N. Immerman. *Descriptive Complexity*. Graduate Texts in Computer Science. Springer, New York, 1999.

[13] P. Jeavons. On the algebraic structure of combinatorial problems. *Theoretical Computer Science*, 200(1-2):185–204, 1998.

[14] P. Jeavons, D. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44(4):527–548, July 1997.

[15] P. Jeavons, D. Cohen, and J. Pearson. Constraints and universal algebra. *Annals of Mathematics and Artificial Intelligence*, 24:51–67, 1998.

[16] D. Joyce. A classifying invariant of knots, the knot quandle. *Journal of Pure and Applied Algebra*, 23:37–66, 1982.

[17] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum, New York, 1972.

[18] R. W. McGrail and M. Sharac. Correcting the record with knot colorings. Technical Report number BARD-CMSC-2008-04, Bard College, 2008. http://asc.bard.edu/.

[19] R. W. McGrail and M. Sharac. Tricoloring as a corrective measure. In *ACM Communications in Computer Algebra*, volume 42, pages 83–85, 2008.

[20] R. N. McKenzie, G. F. McNulty, and W. F. Taylor. *Algebras, Lattices, and Varieties*, volume I. Wadsworth and Brooks, California, 1987.

[21] U. Montanari. Networks of constraints: Fundamental properties and applications to picture processing. *Information Sciences*, 7:95–132, 1974.

[22] J. H. Przytycki. 3-coloring and other elementary invariants of knots. In *Knot Theory*, volume 42, pages 275–295. Banach Center Publications, 1998.

[23] H. Reidemeister. *Knoten und Gruppen*. Abh. Math. Sem. University of Hamburg, 1926.

[24] D. Rolfsen. *Knots and Links*, volume 7 of *Mathematical Lecture Series*. Publish or Perish, 1976.

[25] T. J. Schaefer. The complexity of satisfiability problems. In *Proceeding of the 10th ACM Symposium on the Theory of Computing (STOC'78)*, pages 216–226, 1978.

[26] R. G. Scharein. *KnotPlot: A Program for Viewing Mathematical Knots*. Centre for Experimental and Constructive Mathematics, Simon Fraser University, October 2002.

[27] M. Sipser. The history and status of the P versus NP question. In *Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC'92)*, pages 603–618, 1992.

[28] A. Szendrei. Term minimal algebras. *Algebra Universalis*, 32(4):439–477, 1994.

[29] J. Wielemaker. *SWI-Prolog 5.6 Reference Manual*, April 2006.