GASSMANN EQUIVALENT DESSINS

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Abstract. A bipartitioned dessin is a pair of permutations \((\sigma_0, \sigma_1)\) of a finite set. A dessin gives rise to (and is determined by) a graph embedded in a Riemann surface. This paper studies pairs of dessins that arise from Gassmann triples of groups \((G, H, H')\) together with pairs \(g_0, g_1\) of elements in \(G\). We show that the two dessins have isomorphic monodromy groups, have the same branching data and the same number of components. Moreover, the sums of the genera of the components of the two dessins are the same, but we give an example where the individual genera of the components of the first dessin differ from the genera of the components of the second dessin.

1. Introduction

A Gassmann triple \((G, H, H')\) consists of a group and two subgroups that are ”locally conjugate”. The main idea of this paper is to compare pairs of dessins that arise from these triples. In general, we can look at a dessin as being defined by a pair of permutations on a set of edges. We define Gassmann equivalent dessins as being the dessins determined by the permutations that arise when we let two elements \(g_0\) and \(g_1\) in \(G\) act on the coset spaces \(G/H\) and \(G/H'\). Let \(\sigma_0\) (respectively \(\sigma'_0\)) be the permutations of \(G/H\) (respectively \(G/H'\)) arising from \(g_0\). Similarly, let \(\sigma - 1\) (respectively \(\sigma'_1\)) arise from \(g_1\). Define \(\sigma_\infty\) and \(\sigma'_\infty\) by \(\sigma_0 \sigma_1 \sigma_\infty = Id\) and \(\sigma'_0 \sigma'_1 \sigma'_\infty = Id\). We prove that the cycle structure for \(\sigma_j = \sigma_j\) for \(j = 0, 1, \infty\). This result can be formulated as: Gassmann equivalent dessins have the same branching data. The monodromy groups of Gassmann equivalent dessins, the group generated by the two permutations determining the dessin, are isomorphic. Moreover, the number of components of two dessins arising from a Gassmann triple is equal and the sums of genera of the components of the two dessins coincide. Even so, the individual genera lists of the two dessins can be different; we give an example of disconnected dessins where the genera of the components of the first dessin are not the same with the ones of the genera of the second dessin.

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2. Background Material

2.1. Group action on sets.

Definition 2.1. An action of a group $G$ on a finite set $X$ is a group homomorphism $\phi : G \to \text{Aut}(X)$.

For $g \in G$, we have $\phi(g) \in \text{Aut}(X)$. So, $\phi(g)(x) \in X$ for all $x \in X$. For the action of the element $g \in G$ on the element $x \in X$ we will simply use the notation $^gx$ instead of $\phi(g)(x)$.

Alternatively, we can think of the action of a group $G$ on a set $X$ as a mapping $G \times X \to X$ (denoted $(g, x) \to ^gx$) that satisfies the following two conditions:

1. $\epsilon x = x$, for all $x \in X$;
2. $(\sigma \tau)x = \sigma(\tau x)$, for all $\sigma, \tau \in G$ and every $x \in X$.

When we have a group $G$ acting on a set $X$, we call $X$ a $G$-set.

When a group $G$ acts on a set $X$, then $X$ breaks up into a disjoint union of $G$-orbits. For $x \in X$, $\text{Orb}_G(x) = \{^gx, g \in G\} \subset X$.

If there is an $x \in X$ such that $\text{Orb}_G(x) = X$, then we say that $G$ acts transitively on $X$.

Definition 2.2. For $x \in X$, $\text{Stab}_G(x) = \{g \in G \mid ^gx = x\}$.

The stabilizer of $x$ in $G$ is a subgroup of $G$.

Definition 2.3. An isomorphism of $G$-sets $X$, $Y$ is defined as a bijection $\psi : X \to Y$ such that $^g\psi(x) = \psi(^gx)$ for all $x \in X$, $g \in G$. When there is an isomorphism between these $X$ and $Y$ are called isomorphic as $G$-sets, written $X \cong_G Y$.

Example. If $H \subset G$ is a subgroup, let $X = G/H$, with $G$ acting via left-translation of left cosets:

\begin{equation}
^g(g_1H) = gg_1H
\end{equation}

Then $G/H$ is a transitive $G$-set. Every transitive $G$-set is isomorphic to $G/H$ for some subgroup $H \subset G$.

Definition 2.4. The permutation character of an action of the group $G$ on a finite set $X$, also called the fixed point character, is $\chi_X : G \to \{0, 1, 2, \ldots\}$ given by $\chi_X(g) = |\{x \in X \mid ^gx = x\}|$. 

Definition 2.5. Two elements \( a \) and \( b \) in a group \( G \) are called *conjugate* if there is an element \( g \) in \( G \) such that \( gag^{-1} = b \).

Definition 2.6. Two subgroups \( H \) and \( H' \) of the group \( G \) are called *conjugate* if \( H = gH'g^{-1} \) for some \( g \in G \).

Theorem 2.7. The subgroups \( H \) and \( H' \) are conjugate in \( G \) iff the \( G \)-sets \( G/H \) and \( G/H' \) are isomorphic as \( G \)-sets.

Theorem 2.8. Isomorphic \( G \)-sets have the same fixed point character. So, conjugate subgroups give rise to equal fixed point characters.

In this paper, I will mainly be concerned with the action of \( G \) on the cosets \( G/H \). The group \( G \) acts on the coset space \( G/H \) by left multiplication, i.e. \( \cdot bH = abH \). Let me summarize the main points. The coset action is transitive. Moreover, \( G/H \cong G/H' \) iff \( H \) is conjugate to \( H' \). If \( u \in G \) is any element for which \( uH'u^{-1} = H \), then the map \( \psi : G/H \to G/H' \) defined by \( \psi(gH) = (gH)u = g(uH'u^{-1})u = (gu)H' \) is a well-defined bijection from \( G/H \) to \( G/H' \) that respects the \( G \)-action and induces an isomorphism of \( G \)-sets. So, \( G/H \) and \( G/H' \) have equal fixed point characters.

2.2. Gassmann triples.

Definition 2.9. Let \( G \) be a group and let \( H \) and \( H' \) be two subgroups of \( G \). \((G, H, H')\) is a Gassmann triple if \( \chi_{G/H}(g) = \chi_{G/H'}(g) \) for all \( g \in G \).

Requiring \( \chi_{G/H}(g) = \chi_{G/H'}(g) \) is equivalent to saying that the subgroups \( H, H' \) are locally conjugate, i.e. there is a set bijection \( b : H \to H' \) where \( b(h) \) is conjugate to \( h \) in \( G \) for all \( h \in H \).

The triple \((G, H, H')\) is said to be trivial if \( H \) and \( H' \) are conjugate in \( G \).

Lemma 2.10. If \((G, H, H')\) is a Gassmann triple, then \( |G/H| = |G/H'| \), so \( (G : H) = (G : H') \), so \( |H| = |H'| \).

2.3. Dessins.

Definition 2.11. A *bipartitioned graph* is a graph with a fixed two coloring of the vertices (black, white) such that every edge connects two vertices of different colors.
Definition 2.12. A bipartitioned dessin d’enfant, or dessin for short, is a bipartitioned graph with a cyclic counterclockwise ordering given to the set of edges meeting at each vertex. Work of Grothendieck shows that, equivalently, a dessin is a finite bipartitioned graph together with an embedding into a compact oriented Riemann surface.

One way to get a dessin is to choose an ordered pair \( \sigma_0, \sigma_1 \) in the symmetric group \( S_n \). Given the pair \( \sigma_0, \sigma_1 \), we create the dessin as follows: draw a white vertex for each cycle in \( \sigma_0 \). If a cycle in \( \sigma_0 \) is \((n_1, \ldots, n_r)\), then draw \( r \) half-edges sprouting from the corresponding white vertex and label the half-edges counterclockwise \( n_1, \ldots, n_r \). Similarly draw a black vertex for each cycle in \( \sigma_1 \) with half-edges labeled (counterclockwise) by the elements in that cycle. Finally connect the half-edges with the same label. This produces a bipartitioned graph with a cyclic counterclockwise ordering of the edges emanating from each vertex. Each cycle in \( \sigma_0 \) represents the cyclic ordering of the edges meeting at a white vertex, and each cycle in \( \sigma_1 \) represents the cyclic ordering of the edges meeting at a black vertex. Note that the cycles in \( \sigma_0 \) can be recovered from the dessin by writing down the edge ordering at each of the white vertices, and \( \sigma_1 \) can be similarly recovered by writing down the edge ordering at each of the black vertices.

Definition 2.13. Two dessins \( \sigma_0, \sigma_1 \) and \( \sigma'_0, \sigma'_1 \) are isomorphic if there is \( \tau \in S_n \) that simultaneously conjugates \( \sigma_j \) to \( \sigma'_j \) for \( j = 0, 1 \).

Usually one also requires a dessin to be a connected graph. For this paper we allow dessins to have several components.

Definition 2.14. Let \( G \) be a group and \( H \subset G \) a subgroup of finite index and let \( g_0, g_1 \) be two elements in \( G \). The dessin \( \mathcal{D}(G/H, g_0, g_1) \) is the dessin determined by the permutations \( \sigma_0 \) and \( \sigma_1 \) that are defined by the left \( G \)-action of \( g_0 \) and \( g_1 \) on \( G/H \).

Definition 2.15. Gassmann equivalent dessins are bipartitioned dessins \( \mathcal{D}(G/H, g_0, g_1) \) and \( \mathcal{D}(G/H', g_0, g_1) \) that arise from two chosen elements \( g_0 \) and \( g_1 \) in \( G \) acting by left multiplication on the coset spaces \( G/H \) and \( G/H' \) where \( (G, H, H') \) is a Gassmann triple.

Example. Let \( G \) be the simple group of order 168.\(^1\) There are two subgroups of index 7, \( H \) and \( H' \), and together with \( G \) they form a Gassmann triple. (Professor Robert Perlis showed that this is the only Gassmann triple where the subgroups have index 7 and that there are

\(^1\)It is known that there is only one simple group of order 168. See literature for proof.
none where the index of the subgroups is less than 7). Magma gives us a presentation of this group as generated by 3 elements \(x, y\) and \(z\) that satisfy the following relations:

\[
\begin{align*}
x^2 &= Id \\
y^3 &= Id \\
z &= xy \\
z^7 &= Id
\end{align*}
\]

If we let \(x\) and \(y\) act on the set of 7 cosets \(G/H\), we get the permutations \(\sigma_0 = (1)(2 4)(3)(5)(6 7)\) and \(\sigma_1 = (1 2 3)(4 5 6)(7)\), and if we let \(x\) and \(y\) act on the set of 7 cosets \(G/H'\), we get the permutations \(\sigma'_0 = (1)(2 4)(3)(5 7)(6)\) and \(\sigma'_1 = (1 2 3)(4 5 6)(7)\). The two corresponding dessins for these pairs of permutations are:

![Figure 1. Two non-isomorphic Gassmann equivalent dessins](image)

Each edge stands for a coset and each vertex is determined by the pairs permutations \(\sigma_0, \sigma_1, \sigma'_0, \sigma'_1\) in the way described above.

3. Branching data of Gassmann equivalent dessins

The work of Grothendieck and Belyi showed how dessins correspond to coverings of the 2-sphere with ramification only above three points ("0", "1" and "\(\infty\)") on the 2-sphere.

Given a dessin \(\sigma_0, \sigma_1\), define \(\sigma_\infty\) by the relation \(\sigma_0 \sigma_1 \sigma_\infty = 1\). Then, \(\sigma_\infty\) is determined by \(\sigma_0\) and \(\sigma_1\). When we look at the cycles that determine a dessin, we can read off the branching data in the following way:

For every \(i\)-cycle in \(\sigma_0\) there are \(i\) sheets coming in above 0.

For every \(i\)-cycle in \(\sigma_1\) there are \(i\) sheets coming in above 1.

For every \(i\)-cycle in \(\sigma_\infty\) there are \(i\) sheets coming in above \(\infty\).
Theorem 3.1. If \((G, H, H')\) is a Gassmann triple, then the branching data for \(\mathbb{D}(G/H, g_0, g_1)\) and \(\mathbb{D}(G/H', g_0, g_1)\) coincide.

Proof. Let \(\sigma_0\) be the permutation of \(G/H\) coming from the left multiplication by \(g_0\) and \(\sigma'_0\) be the permutation of \(G/H'\) coming from the left multiplication by \(g_0\). Similarly define \(\sigma_1\) and \(\sigma'_1\) as coming from the action of \(g_1\) on \(G/H\) and \(G/H'\). The branching data of \(\mathbb{D}(G/H, g_0, g_1)\) and \(\mathbb{D}(G/H', g_0, g_1)\) coincide.

Since \((G, H, H')\) is a Gassmann triple, we know that \(\chi_{G/H}(x) = \chi_{G/H'}(x)\) for all \(x \in G\).

The action of \(G\) on \(G/H\) and \(G/H'\) give two homomorphisms \(\phi: G \to \text{Aut}(G/H)\) and \(\phi': G \to \text{Aut}(G/H')\) defined by \(\phi(g) = \sigma\) and \(\phi'(g) = \sigma'\) where \(\sigma\) is the permutation that permutes the cosets of \(H\) when \(g\) acts on \(G/H\) and \(\sigma'\) is the permutation that permutes the cosets of \(H'\) when \(g\) acts on \(G/H'\).

Since \(\phi\) and \(\phi'\) are homomorphisms,

\[
\phi(g^i) = \sigma^i \quad \text{and} \quad \phi'(g^i) = \sigma'^i.
\]

Also,

\[
\chi_{G/H}(g^i) = \chi_{G/H'}(g^i), \forall i \in \mathbb{Z}_+.
\]

Let \(c_j\) be the number of \(j\)-cycles in \(\sigma_0\) and let \(d_j\) be the number of \(j\)-cycles in \(\sigma'_0\).

\[
\chi_{G/H}(g) = c_1 = \chi_{G/H'}(g) = d_1
\]

\[
\chi_{G/H}(g^2) = c_1 + 2c_2 = \chi_{G/H'}(g^2) = d_1 + 2d_2
\]

\[
\vdots
\]

\[
\vdots
\]

Inductively, for every non-negative integer \(i\),

\[
\chi_{G/H}(g^i) = \sum_{k=1}^{i} k c_k = \chi_{G/H'}(g^i) = \sum_{k=1}^{i} k d_k.
\]

Combining the equations, we get that \(c_i = d_i\) for all \(i \in \mathbb{Z}_+\).
This means that $\sigma$ and $\sigma'$ have the same cycle structure for any element $g \in G$ that acts on the set of cosets of $G/H$ and $G/H'$.

As a result, if we take a Gassmann triple $(G, H, H')$ and if we let two elements $g_0$ and $g_1$ act on $G/H$ and $G/H'$ and define $\sigma_0$, $\sigma_1$, $\sigma'_0$ and $\sigma'_1$ by:

$$\phi(g_0) = \sigma_0 \text{ and } \phi(g_1) = \sigma_1,$$

$$\phi'(g_0) = \sigma'_0 \text{ and } \phi'(g_1) = \sigma'_1,$$

then $\sigma_0$ and $\sigma'_0$ have the same cycle structure, and $\sigma_1$ and $\sigma'_1$ have the same cycle structure.

Also, in general, $\sigma_{\infty} = \sigma_1^{-1}\sigma_0^{-1}$, so $\sigma_{\infty}$ is the permutation that arises if we let the element $g_1^{-1}g_0^{-1}$ act on the set of cosets.

So, since for any $g_1$ and $g_2$ we choose, $\chi_{G/H}(g_1^{-1}g_0^{-1}) = \chi_{G/H'}(g_1^{-1}g_0^{-1})$, by using the same argument as above, we get $\sigma_{\infty}$ and $\sigma'_{\infty}$ have the same cycle structure.

**Conclusion**

Since the three permutations that determine two dessins $\mathbb{D}, \mathbb{D}'$ arising from a Gassmann triple have the same cycle structure, the two dessins have the same branching data. This implies that they have the same number of vertices of each color. Moreover, the white (black) vertices of $\mathbb{D}$ can be matched with the white (black) vertices of $\mathbb{D}'$, so that corresponding white (black) vertices have the same number of edges coming in.

**Corollary 3.2.** If we let $g_0 = g_1$, the resulting Gassmann equivalent dessins are isomorphic.

**Proof.** If $g_0 = g_1$, then $\sigma_0 = \sigma_1$ and $\sigma'_0 = \sigma'_1$. Since by theorem 3.1 $\sigma_0$ and $\sigma'_0$ have the same cycle structure, $\exists \tau \in S_n$ such that $\tau \sigma_0 \tau^{-1} = \sigma'_0$. So also, $\tau \sigma_1 \tau^{-1} = \sigma'_1$. So, by definition 2.13, $\mathbb{D}(G/H, g_0, g_1) \cong \mathbb{D}(G/H', g_0, g_1)$.

4. **Monodromy Groups**

**Definition 4.1.** The monodromy group of a dessin is the permutation group $\langle \sigma_0, \sigma_1 \rangle$ generated by the two permutations $\sigma_0$ and $\sigma_1$ that determine the dessin.

**Lemma 4.2.** Let $(G, H, H')$ be a Gassmann triple. The kernels $\text{Ker}_{\phi}$ and $\text{Ker}_{\phi'}$ of the two homomorphisms $\phi : G \to \text{Aut}(G/H)$ and $\phi' : G \to \text{Aut}(G/H')$ determined by the action of $G$ on $G/H$ and $G/H'$ are the same.
Proof. The elements in the kernels of $\phi$ and $\phi'$ leave all the cosets fixed, so:

\[ \text{Ker}_\phi = \{ g \in G | \chi_{G/H}(g) = (G : H) \} \]

\[ \text{Ker}_{\phi'} = \{ g \in G | \chi_{G/H'}(g) = (G : H') \} \]

Since $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$, $\text{Ker}_\phi = \text{Ker}_{\phi'}$. □

**Theorem 4.3.** If $(G, H, H')$ is a Gassmann triple the monodromy groups of $D(G/H, g_0, g_1)$ and $D(G/H', g_0, g_1)$ are isomorphic.

Proof. Let $(G, H, H')$ be a Gassmann triple. Then $G$ acts on the sets of cosets $G/H$ and $G/H'$ giving two group homomorphisms $\phi : G \to \text{Aut}(G/H)$ and $\phi' : G \to \text{Aut}(G/H')$.

Let $M = \langle g_0, g_1 \rangle$ be the subgroup of $G$ generated by $g_0$ and $g_1$ are in $G$. The monodromy groups are by definition, $\phi(M) = \langle \sigma_0, \sigma_1 \rangle$ and $\phi'(M) = \langle \sigma'_0, \sigma'_1 \rangle$. Since $\phi$ and $\phi'$ are homomorphic maps from $G$ to Aut$(G/H)$ and Aut$(G/H')$ and since $M$ is a subgroup of $G$, by the fundamental theorem of homomorphism, $\phi(M)$ and $\phi'(M)$ are isomorphic to $M/\text{Ker}_\phi \cap M$, respectively $M/\text{Ker}_{\phi'} \cap M$. Since $\text{Ker}_\phi$ and $\text{Ker}_{\phi'}$ are the same by lemma 4.2, $\phi(M)$ and $\phi'(M)$ are isomorphic to each other. □

**Example.** Let $G$ be the simple group of order 168 and, as we already pointed out, there are two subgroups of index 7, $H$ and $H'$, and together with $G$ they form a Gassmann triple.

Let $\phi : G \to \text{Aut}(G/H)$ and $\phi' : G \to \text{Aut}(G/H')$ be the homomorphisms determined by the action of $G$ on $G/H$ and $G/H'$. Let $K_\phi$ be the kernel of $\phi$. Then, $K_\phi$ is a normal subgroup of $G$. Since $G$ is simple, the only two normal subgroups of $G$ are $\{e\}$ where $e$ is the identity, and $G$ itself.

Since $G$ acts transitively on the seven cosets $G/H$, there is $g \in G$ that does not like the identity. So, $K_\phi \neq G$. So, $K_\phi = \{e\}$. Since $K_\phi = \{e\}$, $\phi$ is injective. Analogously, we show $\phi'$ is injective.

So, $\phi(M) \cong M$ and $\phi'(M) \cong M$ for $M \subset G$. So, the monodromy groups $\langle \sigma_0, \sigma_1 \rangle = \phi(M)$ and $\langle \sigma'_0, \sigma'_1 \rangle = \phi'(M)$ are isomorphic, independent of how $g_0$ and $g_1$ in $G$ are chosen.

5. **Components and Genera**

5.1. **Components.** Before we prove the following lemma, we specify that saying that $\langle g_0, g_1 \rangle$ acts transitively on a set $X$ is equivalent to saying that $\langle \sigma_0, \sigma_1 \rangle$ acts transitively on $X$ (because we can look at the group Aut$(X)$ as acting on $X$).
Lemma 5.1. Let $X = \{e_1, e_2, \ldots, e_n\}$ be the edge set of a dessin. Then $<\sigma_0, \sigma_1 > \subset S_n$ acts transitively on $X$ iff the dessin is connected.

Proof. Suppose $<\sigma_0, \sigma_1 >$ acts transitively on $X$. Then, $\forall e_i, e_j \in X, \exists \sigma \in <\sigma_0, \sigma_1 >$ such that $\sigma e_i = e_j$. Each $e_i \in X$ is an edge of the dessin, and we define the action of $\sigma_0$ on $e_i$ by a counterclockwise rotation of the edge $e_i$ onto the next edge and the action of $\sigma_0^{-1}$ on $e_i$ by a clockwise rotation of the edge $e_i$ onto the next edge at a white vertex. Similarly, we define the actions of $\sigma_1$ and $\sigma_1^{-1}$ by rotations at a black vertex. So, geometrically, $\sigma e_i = e_j$ means that by a series of rotations of the edge $e_i$ we can get to the edge $e_j$. Since this is true for all $e_i, e_j \in X$, the dessin is connected.

Also, if the dessin is connected, we can get by a series of rotations from the edge $e_i$ to the edge $e_j$ for all $e_i, e_j$ in $X$.

Example. Let $\sigma_0 = (1 \ 2 \ 3)$ and $\sigma_1 = (1 \ 2)(3)$. The dessin resulting from $\sigma_0$ and $\sigma_1$ is:

![Figure 2. Connected dessin](image)

$<\sigma_0, \sigma_1 >$ acts transitively on the set $\{e_1, e_2, e_3\}$. We are explicitly going to show how from every edge one can get any other edge:

- $e_1 = \sigma_1 e_2$
- $e_1 = \sigma_0^2 e_3 = \sigma_0^{-1} e_3$
- $e_2 = \sigma_1 e_1$
- $e_2 = \sigma_0 e_3$
- $e_3 = \sigma_0 e_1$
- $e_3 = \sigma_0^2 e_1 = \sigma_0^{-1} e_2$

Corollary 5.2. The number of components of a dessin is the number of orbits of $<\sigma_0, \sigma_1 >$ on $X$. 
Proof. The action of each orbit of $\langle \sigma_0, \sigma_1 \rangle$ under $\{e_1, e_2, ..., e_n\}$ is transitive, so each orbit determines a connected dessin. □

Example. Let $\sigma_0 = (1 \ 2 \ 3)(4 \ 5 \ 6)(7)$ and $\sigma_1 = (1 \ 3 \ 2)(4 \ 6 \ 5)(7)$. Then the monodromy group $M = \langle \sigma_0, \sigma_1 \rangle$ fixes 7, so $M$ is not transitive on $X = \{e_1, e_2, ..., e_7\}$. $X$ splits into 3 orbits (3 different $G$-sets): $X_1 = \{e_1, e_2, e_3\}$, $X_2 = \{e_4, e_5, e_6\}$ and $X_3 = \{e_7\}$. The corresponding dessin is disconnected and has 3 components:

![Disconnected dessin](image)

**Figure 3.** Disconnected dessin

**Theorem 5.3.** If $(G, H, H')$ is a Gassmann triple, then $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ have the same number of components.

Proof. Let $(G, H, H')$ be a Gassmann triple. Let the fixed point characters of the action of $G$ on $G/H$ and $G/H'$ be $\chi$, respectively $\chi'$. So, $\chi = \chi'$.

Let $< g_0, g_1 > \subset G$, so $\chi|_{<g_0,g_1>} = \chi'|_{<g_0,g_1>}$. Let $o$ be the number of orbits of $< g_0, g_1 >$ on $G/H$ and $o'$ be the number of orbits of $< g_0, g_1 >$ on $G/H'$. By Burnside’s formula,

$$
o = \frac{1}{\langle g_0, g_1 \rangle} \sum_{g \in \langle g_0, g_1 \rangle} \chi|_{<g_0,g_1>} (g);
$$

$$
o' = \frac{1}{\langle g_0, g_1 \rangle} \sum_{g \in \langle g_0, g_1 \rangle} \chi'|_{<g_0,g_1>} (g).
$$

Since $\chi|_{<g_0,g_1>} = \chi'|_{<g_0,g_1>}$, $o = o'$. So, by corollary 5.2, $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ have the same number of components. □

**Corollary 5.4.** $< \sigma_0, \sigma_1 >$ is transitive on $G/H \iff < \sigma'_0, \sigma'_1 >$ is transitive on $G/H'$;
Proof. If $<\sigma_0,\sigma_1>$ is transitive on $G/H$, then $o = 1$, so $o' = 1$. So, $<\sigma_0',\sigma_1'>$ is transitive on $G/H'$.
Analagoulsly, if $<\sigma_0,\sigma_1>$ is transitive on $G/H'$, then $<\sigma_0,\sigma_1>$ is transitive on $G/H$. □

5.2. Genera.

Theorem 5.5. If $(G,H,H')$ is a Gassmann triple, then $\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \gamma_i'$ where $k$ is the number of components of $D(G/H,g_0,g_1)$ and $D(G/H',g_0,g_1)$, $\gamma_i$ denotes the genus of the $i$-th component of $D(G/H,g_0,g_1)$ and $\gamma_i'$ denotes the genus of the $i$-th component of $D(G/H',g_0,g_1)$

Proof. We compute the genus of a connected dessin by using the Riemann-Hurwitz equation

$$2 - 2\gamma = 2n - \sum_{i=1}^{k}(e_p - 1)$$

where $\gamma$=genus, $n$=number of edges and $e_p$=(number of edges coming into a branch point).

Let $n_i$ and $n_i'$ denote the numbers of edges of the $i$-th component of $D(G/H,g_0,g_1)$ and of $D(G/H',g_0,g_1)$ respectively, and let $e_{p_i}$ and $e_{p_i}'$ denote the numbers of edges coming into a branch point of the $i$-th component of $D(G/H,g_0,g_1)$ and of $D(G/H',g_0,g_1)$ respectively.

So, applying the Riemann-Hurwitz formula to each component and summing over $k$ we get:

$$2k - 2 \sum_{i=1}^{k} \gamma_i = 2 \sum_{i=1}^{k} n_i - \sum_{i=1}^{k}(e_{p_i} - 1)$$

and

$$2k - 2 \sum_{i=1}^{k} \gamma_i' = 2 \sum_{i=1}^{k} n_i' - \sum_{i=1}^{k}(e_{p_i}' - 1).$$

Since the number of edges of $D(G/H,g_0,g_1)$ is $|G/H|$, the number of edges of $D(G/H',g_0,g_1)$ is $|G/H'|$ and $|G/H| = |G/H'|$, it follows that $\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} n_i'$. Also, according to theorem 3.1, the number of edges coming into each branch point is the same for the two dessins, so $\sum_{i=1}^{k}(e_{p_i} - 1) = \sum_{i=1}^{k}(e_{p_i}' - 1)$.

It then follows comparing the two equations that $\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} \gamma_i'$.

□

Corollary 5.6. If $<\sigma_0,\sigma_1>$ is transitive on $G/H$, $\gamma = \gamma'$ where $\gamma$ is the genus of $D(G/H,g_0,g_1)$ and $\gamma'$ is the genus of $D(G/H',g_0,g_1)$.

Proof. By corollary 5.4, since $<\sigma_0,\sigma_1>$ is transitive on $G/H$, $<\sigma_0',\sigma_1'>$ is also transitive on $G/H'$, so by lemma 5.1 $D(G/H,g_0,g_1)$
and $\mathbb{D}(G/H', g_0, g_1)$ are connected. Since each dessin has just one component, if we use the Riemann-Hurwitz equation, we get $\gamma = \gamma'$. □

In the case of disconnected Gassmann equivalent dessins, even though the sum of the genera of the components of the first dessin is always equal to the sum of the genera of the second dessin, the individual genera of the components of the first dessin might differ from the ones of the components of the second dessin.

**Example.** Let $G = GL_2(\mathbb{F}_5)$.

Let $H = \{ \begin{pmatrix} a^2 & x \\ 0 & c \end{pmatrix} \mid a^2c \neq 0 \}$ and $H' = \{ \begin{pmatrix} c & x \\ 0 & a^2 \end{pmatrix} \mid a^2c \neq 0 \}$. 

$(G, H, H')$ is a Gassmann triple of index 12.

Let $g_0 = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}$ and $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

If we let $g_0$ and $g_1$ act on $G/H$ we get the permutations 
$\sigma_0 = (1 \; 3 \; 7 \; 4)(2 \; 6 \; 5 \; 8)(9)(10 \; 12)(11)$ and $\sigma_1 = (1 \; 7 \; 3 \; 11 \; 4)(2 \; 9 \; 5 \; 8 \; 6)(10)(12)$.

If we let $g_0$ and $g_1$ act on $G/H'$ we get the permutations 
$\sigma'_0 = (1 \; 3 \; 10 \; 4)(2 \; 7 \; 12 \; 9)(6)(8)(5 \; 11)$ and $\sigma'_1 = (1 \; 5 \; 10 \; 7 \; 9)(2 \; 4 \; 3 \; 12 \; 11)(6)(8)$.

The two resulting dessins are the following:

![Figure 4](image_url)

**Figure 4.** $\mathbb{D}(G/H; g_0, g_1)$

$\gamma_1 + \gamma_2 + \gamma_3 = 1 + 1 + 0 = 2$. 
Figure 5. $\mathbb{D}(G/H', g_0, g_1)$
$\gamma'_1 + \gamma'_2 + \gamma'_3 = 1 + 1 + 0 = 2.$