GASSMANN EQUIVALENT DESSINS

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ABSTRACT. A *bipartitioned dessin* is a pair of permutations (σ_0, σ_1) of a finite set. A dessin gives rise to (and is determined by) a graph embedded in a Riemann surface. This paper studies pairs of dessins that arise from Gassmann triples of groups (G, H, H') together with pairs g_0, g_1 of elements in G. We show that the two dessins have isomorphic monodromy groups, have the same branching data and the same number of components. Moreover, the sums of the genera of the components of the two dessins are the same, but we give an example where the individual genera of the components of the first dessin differ from the genera of the components of the second dessin.

1. INTRODUCTION

A Gassmann triple (G, H, H') consists of a group and two subgroups that are "locally conjugate". The main idea of this paper is to compare pairs of dessins that arise from these triples. In general, we can look at a dessin as being defined by a pair of permutations on a set of edges. We define Gassmann equivalent dessins as being the dessins determined by the permutations that arise when we let two elements g_0 and g_1 in G act on the coset spaces G/H and G/H'. Let σ_0 (respectively σ'_0) be the permutations of G/H (respectively G/H') arising from g_0. Similarly, let $\sigma - 1$ (respectively σ'_1) arise from g_1. Define σ_∞ and σ'_∞ by $\sigma_0\sigma_1\sigma_\infty = Id$ and $\sigma'_0\sigma'_1\sigma'_\infty = Id$. We prove that the cycle structure for $\sigma_j = \sigma'_j$ for $j = 0, 1, \infty$. This result can be formulated as: Gsaamann equivalent dessins have the same branching data. The monodromy groups of Gassmann equivalent dessins, the group generated by the two permutations determining the dessin, are isomorphic. Moreover, the number of components of two dessins arising from a Gassmann triple is equal and the sums of genera of the components of the two dessins coincide. Even so, the individual genera lists of the two dessins can be different; we give an example of disconnected dessins where the genera of the components of the first dessin are not the same with the ones of the genra of the second dessin.

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2. Background Material

2.1. Group action on sets.

Definition 2.1. An *action* of a group G on a finite set X is a group homomorphism $\phi : G \to \mathrm{Aut}(X)$.

For $g \in G$, we have $\phi(g) \in \mathrm{Aut}(X)$. So, $\phi(g)(x) \in X$ for all $x \in X$. For the action of the element $g \in G$ on the element $x \in X$ we will simply use the notation ${}^{g}x$ instead of $\phi(g)(x)$.
Alternatively, we can think of the action of a group G on a set X as a mapping $G \times X \to X$ (denoted $(g, x) \to {}^{g}x$) that satisties the following two conditions:

(1) ${}^{e}x = x$, for all $x \in X$;

(2) ${}^{(\sigma\tau)}x = {}^{\sigma}({}^{\tau}x)$, for all $\sigma, \tau \in G$ and every $x \in X$.

When we have a group G acting on a set X, we call X a $G-set$.
When a group G acts on a set X, then X breaks up into a disjoint union of $G-orbits$. For $x \in X$, $Orb_G(x) = \{{}^{g}x, g \in G\} \subset X$.
If there is an $x \in X$ such that $Orb_G(x) = X$, then we say that G acts transitively on X.

Definition 2.2. For $x \in X, Stab_G(x) = \{g \in G \mid {}^{g}x = x\}$.

The stabilizer of x in G is a subgroup of G.

Definition 2.3. An *isomorphism of* $G-sets$ X, Y is defined as a bijection $\psi : X \to Y$ such that ${}^{g}\psi(x) = \psi({}^{g}x)$ for all $x \in X$, $g \in G$. When there is an isomorphism between these X and Y are called *isomorphic as G-sets*, written $X \cong_G Y$.

Example. If $H \subset G$ is a subgroup, let $X = G/H$, with G acting via left-translation of left cosets:

$$ {}^{g}(g_1 H) = g g_1 H \tag{2.1} $$

Then G/H is a transitive $G-set$. Every transitive $G-set$ is isomorphic to G/H for some subgroup $H \subset G$.

Definition 2.4. The *permutation character* of an action of the group G on a finite set X, also called the *fixed point character*, is $\chi_X : G \to \{0, 1, 2, ...\}$ given by $\chi_X(g) = |\{x \in X \mid {}^{g}x = x\}|$.

Definition 2.5. Two elements a and b in a group G are called *conjugate* if there is an element g in G such that $gag^{-1} = b$.

Definition 2.6. Two subgroups H and H' of the group G are called *conjugate* if $H = gH'g^{-1}$ for some $g \in G$.

Theorem 2.7. *The subgroups H and H' are conjugate in G iff the $G-$sets G/H and G/H' are isomorphic as $G-$sets.*

Theorem 2.8. *Isomorphic $G-$sets have the same fixed point character. So, conjugate subgroups give rise to equal fixed point characters.*

In this paper, I will mainly be concerned with the action of G on the cosets G/H. The group G acts on the coset space G/H by left multiplication, i.e. ${}^{a}bH = abH$. Let me summarize the main points. The coset action is transitive. Moreover, $G/H \cong_G G/H'$ iff H is conjugate to H'. If $u \in G$ is any element for which $uH'u^{-1} = H$, then the map $\psi : G/H \to G/H'$ defined by $\psi(gH) = (gH)u = g(uH'u^{-1})u = (gu)H'$ is a well-defined bijection from G/H to G/H' that respects the *G–action* and induces an isomorphism of $G-sets$. So, G/H and $G/H\prime$ have equal fixed point characters.

2.2. **Gassmann triples.**

Definition 2.9. Let G be a group and let H and H' be two subgroups of G. (G, H, H') is a Gassmann triple if $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$.

Requiring $\chi_{G/H}(g) = \chi_{G/H'}(g)$ is equivalent to saying that the subgroups H, H' are *locally conjugate*, i.e. there is a set bijection $b : H \to H'$ where $b(h)$ is conjugate to h in G for all $h \in H$.

The triple (G, H, H') is said to be trivial if H and H' are conjugate in G.

Lemma 2.10. *If (G, H, H') is a Gassmann triple, then $|G/H| = |G/H'|$, so $(G : H) = (G : H')$, so $|H| = |H'|$.*

2.3. **Dessins.**

Definition 2.11. A *bipatitioned graph* is a graph with a fixed two coloring of the vertices (black, white) such that every edge connects two vertices of different colors.

Definition 2.12. A *bipartitioned dessin d'enfant*, or *dessin* for short, is a bipartitioned graph with a cyclic counterclockwise ordering given to the set of edges meeting at each vertex. Work of Grothendieck shows that, equivalently, a dessin is a finite bipartitioned graph together with an embedding into a compact oriented Riemann surface.

One way to get a dessin is to choose an ordered pair σ_0, σ_1 in the symmetric group S_n. Given the pair σ_0, σ_1, we create the dessin as follows: draw a white vertex for each cycle in σ_0. If a cycle in σ_0 is $(n_1, .., n_r)$, then draw r half-edges sprouting from the corresponding white vertex and label the half-edges counterclockwise $n_1, .., n_r$. Similarly draw a black vertex for each cycle in σ_1 with half-edges labeled (counterclockwise) by the elements in that cycle. Finally connect the half-edges with the same label. This produces a bipatitioned graph with a cyclic counterclockwise ordering of the edges emanating from each vertex. Each cycle in σ_0 represents the cyclic ordering of the edges meeting at a white vertex, and each cycle in σ_1 represents the cyclic ordering of the edges meeting at a black vertex. Note that the cycles in σ_0 can be recovered from the dessin by writing down the edge ordering at each of the white vertices, and σ_1 can be similarly recovered by writing down the edge ordering at each of the balck vertices.

Definition 2.13. Two dessins σ_0, σ_1 and σ_0', σ_1' are *isomorphic* if there is $\tau \in S_n$ that simultaneously conjugates σ_j to σ_j' for $j = 0, 1$.

Usually one also requires a dessin to be a connected graph. For this paper we allow dessins to have several components.

Definition 2.14. Let G be a group and $H \subset G$ a subgroup of finite index and let g_0, g_1 be two elements in G. The *dessin* $\mathbb{D}(G/H, g_0, g_1)$ is the dessin determined by the permutations σ_0 and σ_1 that are defined by the left G-action of g_0 and g_1 on G/H.

Definition 2.15. *Gassmann equivalent dessins* are bipartitioned dessins $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ that arise from two chosen elements g_0 and g_1 in G acting by left multiplication on the coset spaces G/H and G/H' where (G, H, H') is a Gassmann triple.

Example. Let G be the simple group of order 168.[1] There are two subgroups of index 7, H and H', and together with G they form a Gassmann triple. (Professor Robert Perlis showed that this is the only Gassmann triple where the subgroups have index 7 and that there are

[1]It is known that there is only one simple group of order 168. See literature for proof.

none where the index of the subgroups is less than 7). Magma gives us a presentation of this group as generated by 3 elements x, y and z that satisfy the following relations:

$$\begin{aligned} x^2 &= Id \\ y^3 &= Id \\ z &= xy \\ z^7 &= Id \end{aligned}$$

If we let x and y act on the set of 7 cosets G/H, we get the permutations $\sigma_0 = (1)(2\ 4)(3)(5)(6\ 7)$ and $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6)(7)$, and if we let x and y act on the set of 7 cosets G/H', we get the permutations $\sigma_0' = (1)(2\ 4)(3)(5\ 7)(6)$ and $\sigma_1' = (1\ 2\ 3)(4\ 5\ 6)(7)$. The two corresponding dessins for these pairs of permutations are:

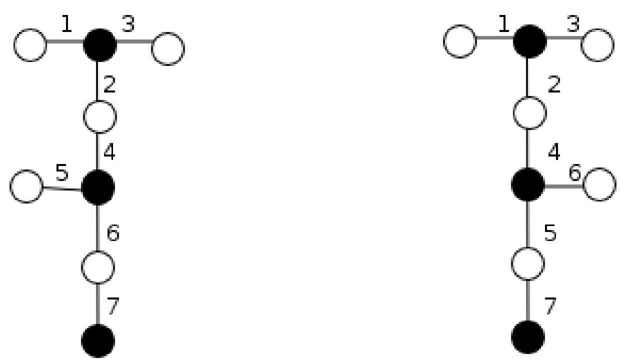

FIGURE 1. Two non-isomorphic Gassmann equivalent dessins

Each edge stands for a coset and each vertex is determined by the pairs permutations σ_0, σ_1, and σ_0', σ_1' in the way described above.

3. Branching data of Gassmann equivalent dessins

The work of Grothendieck and Belyi showed how dessins correspond to coverings of the 2-sphere with ramification only above three points ("0", "1" and "∞") on the 2-sphere.
Given a dessin σ_0, σ_1, define σ_∞ by the relation $\sigma_0\sigma_1\sigma_\infty = 1$. Then, σ_∞ is determined by σ_0 and σ_1. When we look at the cycles that determine a dessin, we can read off the branching data in the following way:
For every i-cycle in σ_0 there are i sheets coming in above 0.
For every i-cycle in σ_1 there are i sheets coming in above 1.
For every i-cycle in σ_∞ there are i sheets coming in above ∞.

Theorem 3.1. *If (G, H, H') is a Gassmann triple, then the branching data for $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ coincide.*

Proof. Let σ_0 be the permutation of G/H coming from the left multiplication by g_0 and σ'_0 be the permutation of G/H' coming from the left multiplication by g_0. Similarly define σ_1 and σ'_1 as coming from the action of g_1 on G/H and G/H'. The brancching data of $\mathbb{D}(G/H, g_0, g_1)$ is just the cycle structure of σ_0, σ_1 and σ_∞. We want to show that σ_0 has the same cycle structure as σ'_0, σ_1 has the same cycle structure as σ'_1, and σ_∞ has the same cycle structure as σ'_∞.
Since $(G, H, H\prime)$ is a a Gassmann triple, we know that $\chi_{G/H}(x) = \chi_{G/H'}(x)$ for all $x \in G$.
The action of G on G/H and G/H' give two homomorphisms $\phi : G \to \mathrm{Aut}(G/H)$ and $\phi' : G \to \mathrm{Aut}(G/H')$ defined by $\phi(g) = \sigma$ and $\phi'(g) = \sigma'$ where σ is the permutation that permutes the cosets of H when g acts on G/H and σ' is the permutation that permutes the cosets of H' when g acts on G/H'.
Since ϕ and ϕ' are homomorphisms,

$$\phi(g^i) = \sigma^i \text{ and } \phi'(g^i) = \sigma'^i.$$

Also,

$$\chi_{G/H}(g^i) = \chi_{G/H'}(g^i), \forall i \in \mathbb{Z}_+.$$

Let c_j be the number of j-cycles in σ_0 and let d_j be the number of j-cycles in σ'_0.

$$\chi_{G/H}(g) = c_1 = \chi_{G/H'}(g) = d_1$$
$$\chi_{G/H}(g^2) = c_1 + 2c_2 = \chi_{G/H'}(g^2) = d_1 + 2d_2$$

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Inductively, for every non-negative integer i,

$$\chi_{G/H}(g^i) = \sum_{k=1}^{i} kc_k = \chi_{G/H'}(g^i) = \sum_{k=1}^{i} kd_k.$$

Combining the equations, we get that c_i=d_i for all $i \in \mathbb{Z}_+$.

This means that σ and σ' have the same cycle structure for any element $g \in G$ that acts on the set of cosets of G/H and G/H'. □

As a result, if we take a Gassmann triple (G, H, H') and if we let two elements g_0 and g_1 act on G/H and G/H' and define σ_0, σ_1, σ_0' and σ_1' by:

$$\phi(g_0) = \sigma_0 \text{ and } \phi(g_1) = \sigma_1,$$
$$\phi'(g_0) = \sigma_0' \text{ and } \phi'(g_1) = \sigma_1',$$

then σ_0 and σ_0' have the same cycle structure, and σ_1 and σ_1' have the same cycle structure.
Also, in general, $\sigma_\infty = \sigma_1^{-1}\sigma_0^{-1}$, so σ_∞ is the permutation that arises if we let the element $g_1^{-1}g_0^{-1}$ act on the set of cosets.
So, since for any g_1 and g_2 we choose, $\chi_{G/H}(g_1^{-1}g_0^{-1}) = \chi_{G/H'}(g_1^{-1}g_0^{-1})$, by using the same argument as above, we get σ_∞ and σ_∞' have the same cycle structure.

Conclusion
Since the three permutations that determine two dessins $\mathbb{D}, \mathbb{D}'$ arising from a Gassmann triple have the same cycle structure, the two dessins have the same branching data. This implies that they have the same number of vertices of each color. Moreover, the white (black) vertices of $\mathbb{D}$ can be matched with the white (black) vertices of $\mathbb{D}'$, so that corresponding white (black) vertices have the same number of edges coming in.

Corollary 3.2. *If we let $g_0 = g_1$, the resulting Gassmann equivalent dessins are isomorphic.*

Proof. If $g_0 = g_1$, then $\sigma_0 = \sigma_1$ and $\sigma_0' = \sigma_1'$. Since by theorem 3.1 σ_0 and σ_0' have the same cycle structure, $\exists\ \tau \in S_n$ such that $\tau\sigma_0\tau^{-1} = \sigma_0'$. So also, $\tau\sigma_1\tau^{-1} = \sigma_1'$. So, by definition 2.13, $\mathbb{D}(G/H, g_0, g_1) \cong \mathbb{D}(G/H', g_0, g_1)$ □

4. Monodromy Groups

Definition 4.1. The monodromy group of a dessin is the permutation group $< \sigma_0, \sigma_1 >$ generated by the two permutations σ_0 and σ_1 that determine the dessin.

Lemma 4.2. *Let (G, H, H') be a Gassmann triple. The kernels Ker_ϕ and $Ker_{\phi'}$ of the two homomorphisms $\phi : G \to Aut(G/H)$ and $\phi' : G \to Aut(G/H')$ determined by the action of G on G/H and G/H' are the same.*

Proof. The elements in the kernels of ϕ and ϕ' leave all the cosets fixed, so:

$$Ker_\phi = \{g \in G | \chi_{G/H}(g) = (G:H)\}$$

$$Ker_{\phi'} = \{g \in G | \chi_{G/H'}(g) = (G:H')\}$$

Since $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$, $Ker_\phi = Ker_{\phi'}$. □

Theorem 4.3. *If (G, H, H') is a Gassmann triple the monodromy groups of $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ are isomorphic.*

Proof. Let (G, H, H') be a Gassmann triple. Then G acts on the sets of cosets G/H and G/H' giving two group homomorphisms $\phi : G \to \text{Aut}(G/H)$ and $\phi' : G \to \text{Aut}(G/H')$.
Let $M =< g_0, g_1 >$ be the subgroup of G generated by g_0 and g_1 are in G. The monodromy groups are by definition, $\phi(M) =< \sigma_0, \sigma_1 >$ and $\phi'(M) =< \sigma'_0, \sigma'_1 >$. Since ϕ and ϕ' are homomorphic maps from G to $\text{Aut}(G/H)$ and $\text{Aut}(G/H')$ and since M is a subgroup of G, by the fundamental theorem of homomorphism, $\phi(M)$ and $\phi'(M)$ are isomorphic to $M/Ker_\phi \cap M$, respectively $M/Ker_{\phi'} \cap M$. Since Ker_ϕ and $Ker_{\phi'}$ are the same by lemma 4.2, $\phi(M)$ and $\phi'(M)$ are isomorphic to each other. □

Example. Let G be the simple group of order 168 and, as we already pointed out, there are two subgroups of index 7, H and H', and together with G they form a Gassmann triple.

Let $\phi : G \to \text{Aut}(G/H)$ and $\phi' : G \to \text{Aut}(G/H')$ be the homomorphisms determined by the action of G on G/H and G/H'.
Let K_ϕ be the kernel of ϕ. Then, K_ϕ is a normal subgroup of G. Since G is simple, the only two normal subgroups of G are $\{e\}$ where e is the identity, and G itself.
Since G acts transitively on the seven cosets G/H, there is $g \in G$ that does not like the identity.So, $K_\phi \neq G$. So,$K_\phi = \{e\}$. Since $K_\phi = \{e\}$, ϕ is injective. Analogously, we show ϕ' is injective.
So, $\phi(M) \cong M$ and $\phi'(M) \cong M$ for $M \subset G$. So, the monodromy groups $< \sigma_0, \sigma_1 >= \phi(M)$ and $< \sigma'_0, \sigma'_1 >= \phi'(M)$ are isomorphic, independent of how g_0 and g_1 in G are chosen.

5. Components and Genera

5.1. **Components.** Before we prove the following lemma, we specify that saying that $< g_0, g_1 >$ acts transitively on a set X is equivalent to saying that $< \sigma_0, \sigma_1 >$ acts transitively on X (because we can look at the group $\text{Aut}(X)$ as acting on X).

Lemma 5.1. *Let* $X = \{e_1, e_2, ..., e_n\}$ *be the edge set of a dessin. Then* $< \sigma_0, \sigma_1 > \subset S_n$ *acts transitively on* $X =$ *iff the dessin is connected.*

Proof. Suppose $< \sigma_0, \sigma_1 >$ acts transitively on X. Then, $\forall e_i, e_j \in X$, $\exists\ \sigma \in < \sigma_0, \sigma_1 >$ such that $\sigma e_i = e_j$. Each $e_i \in X$ is an edge of the dessin, and we define the action of σ_0 on e_i by a counterclockwise rotation of the edge e_i onto the next edge and the action of σ_0^{-1} on e_i by a clockwise rotation of the edge e_i onto the next edge at a white vertex. Similarly, we define the actions of σ_1 and σ_1^{-1} by rotations at a black vertex. So, geometrically, $\sigma e_i = e_j$ means that by a series of rotations of the edge e_i we can get to the edge e_j. Since this is true for all $e_i, e_j \in X$, the dessin is connected.
Also, if the dessin is connected, we can get by a series of rotations from the edge e_i to the edge e_j for all e_i, e_j in X. □

Example. Let $\sigma_0 = (1\ 2\ 3)$ and $\sigma_1 = (1\ 2)(3)$. The dessin resulting from σ_0 and σ_1 is:

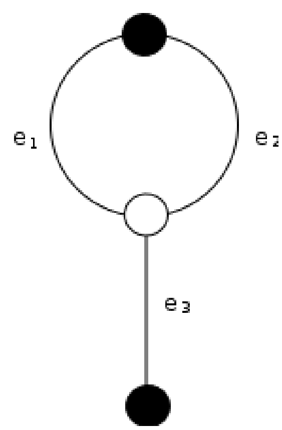

FIGURE 2. Connected dessin

$< \sigma_0, \sigma_1 >$ acts transitively on the set $\{e_1, e_2, e_3\}$. We are explicitly going to show how from every edge one can get any other edge:

$$\begin{aligned} e_1 &= \sigma_1 e_2 \\ e_1 &= \sigma_0^2 e_3 &= \sigma_0^{-1} e_3 \\ e_2 &= \sigma_1 e_1 \\ e_2 &= \sigma_0 e_3 \\ e_3 &= \sigma_0 e_1 \\ e_3 &= \sigma_0^2 e_1 &= \sigma_0^{-1} e_2 \end{aligned}$$

Corollary 5.2. *The number of components of a dessin is the number of orbits of* $< \sigma_0, \sigma_1 >$ *on* X.

Proof. The action of each orbit of $< \sigma_0, \sigma_1 >$ under $\{e_1, e_2, ..., e_n\}$ is transitive, so each orbit determines a connected dessin. □

Example. Let $\sigma_0 = (1\ 2\ 3)(4\ 5\ 6)(7)$ and $\sigma_1 = (1\ 3\ 2)(4\ 6\ 5)(7)$. Then the monodromy group $M =< \sigma_0, \sigma_1 >$ fixes 7, so M is not transitive on $X = \{e_1, e_2, .., e_7\}$. X splits into 3 orbits (3 differents $G - sets$):$X_1 = \{e_1, e_2, e_3\}$, $X_2 = \{e_4, e_5, e_6\}$ and $X_3 = \{e_7\}$. The corresponding dessin is disconnected and has 3 components:

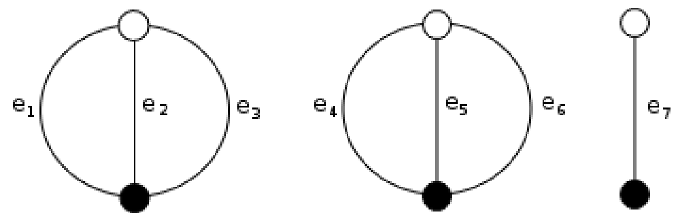

FIGURE 3. Disconnected dessin

Theorem 5.3. *If (G, H, H') is a Gassmann triple, then $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ have the same number of components.*

Proof. Let (G, H, H') be a Gassmann triple. Let the fixed point characters of the action of G on G/H and G/H' be χ, respectively χ'. So, $\chi = \chi'$.
$< g_0, g_1 > \subset G$, so $\chi|_{<g_0,g_1>} = \chi'|_{<g_0,g_1>}$.
Let o be the number of orbits of $< g_0, g_1 >$ on G/H and o' be the number of orbits of $< g_0, g_1 >$ on G/H'. By Burnside's formula,

$$o = \frac{1}{|< g_0, g_1 >|} \sum_{g \in <g_0,g_1>} \chi|_{<g_0,g_1>}(g);$$

$$o' = \frac{1}{|< g_0, g_1 >|} \sum_{g \in <g_0,g_1>} \chi'|_{<g_0,g_1>}(g).$$

Since $\chi|_{<g_0,g_1>} = \chi'|_{<g_0,g_1>}$, $o = o'$. So, by corollary 5.2, $\mathbb{D}(G/H, g_0, g_1)$ and $\mathbb{D}(G/H', g_0, g_1)$ have the same number of components. □

Corollary 5.4. *$< \sigma_0, \sigma_1 >$ is transitive on $G/H \iff < \sigma'_0, \sigma'_1 >$ is transitive on G/H';*

Proof. If $<\sigma_0,\sigma_1>$ is transitive on G/H, then $o=1$, so $o'=1$. So, $<\sigma_0',\sigma_1'>$ is transitive on G/H'.
Analogously, if $<\sigma_0,\sigma_1>$ is transitive on G/H', then $<\sigma_0,\sigma_1>$ is transitive on G/H. □

5.2. **Genera.**

Theorem 5.5. *If (G,H,H') is a Gassmann triple, then $\sum_{i=1}^k \gamma_i = \sum_{i=1}^k \gamma_i'$ where k is the number of components of $\mathbb{D}(G/H,g_0,g_1)$ and $\mathbb{D}(G/H',g_0,g_1)$, γ_i denotes the genus of the i-th component of $\mathbb{D}(G/H,g_0,g_1)$ and γ_i' denotes the genus of the i-th component of $\mathbb{D}(G/H',g_0,g_1)$*

Proof. We compute the genus of a connected dessin by using the Riemann-Hurwitz equation

$$2-2\gamma = 2n - \sum(e_p - 1)$$

where γ=genus, n=number of edges and e_p=(number of edges coming into a branch point).
Let n_i and n_i' denote the numbers of edges of the i-th component of $\mathbb{D}(G/H,g_0,g_1)$ and of $\mathbb{D}(G/H',g_0,g_1)$ respectively, and let e_{p_i} and e_{p_i}' denote the numbers of edges coming into a branch point of the i-th component of $\mathbb{D}(G/H,g_0,g_1)$ and of $\mathbb{D}(G/H',g_0,g_1)$ respectively.
So, applying the Riemann-Hurwitz formula to each component and summing over k we get:

$$\begin{aligned} 2k - 2\sum_{i=1}^k \gamma_i &= 2\sum_{i=1}^k n_i - \sum_{i=1}^k (e_{p_i} - 1) \textit{ and} \\ 2k - 2\sum_{i=1}^k \gamma_i' &= 2\sum_{i=1}^k n_i' - \sum_{i=1}^k (e_{p_i}' - 1). \end{aligned}$$

Since the number of edges of $\mathbb{D}(G/H,g_0,g_1)$ is $|G/H|$, the number of edges of $\mathbb{D}(G/H',g_0,g_1)$ is $|G/H'|$ and $|G/H| = |G/H'|$, it follows that $\sum_{i=1}^k n_i = \sum_{i=1}^k n_i'$. Also, according to theorem 3.1, the number of edges coming into each branch point is the same for the two dessins, so $\sum_{i=1}^k (e_{p_i} - 1) = \sum_{i=1}^k (e_{p_i}' - 1)$.
It then follows comparing the two equations that $\sum_{i=1}^k \gamma_i = \sum_{i=1}^k \gamma_i'$. □

Corollary 5.6. *If $<\sigma_0,\sigma_1>$ is transitive on G/H, $\gamma=\gamma'$ where γ is the genus of $\mathbb{D}(G/H,g_0,g_1)$ and γ' is the genus of $\mathbb{D}(G/H',g_0,g_1)$.*

Proof. By corollary 5.4, since $<\sigma_0,\sigma_1>$ is transitive on G/H, $<\sigma_0',\sigma_1'>$ is also transitive on G/H', so by lemma 5.1 $\mathbb{D}(G/H,g_0,g_1)$

and $\mathbb{D}(G/H', g_0, g_1)$ are connected.
Since each dessin has just one component, if we use the Riemann-Hurwitz equation, we get $\gamma = \gamma'$. □

In the case of disconnected Gassmann equivalent dessins, even though the sum of the genera of the components of the first dessin is always equal to the sum of the genera of the second dessin, the individual genera of the components of the first dessin might differ from the ones of the components of the second dessin.

Example. Let $G = GL_2(\mathbb{F}_5)$.

Let $H = \{\begin{pmatrix} a^2 & x \\ 0 & c \end{pmatrix} \mid a^2c \neq 0\}$ and $H' = \{\begin{pmatrix} c & x \\ 0 & a^2 \end{pmatrix} \mid a^2c \neq 0\}$.

(G, H, H') is a Gassmann triple of index 12.

Let $g_0 = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}$ and $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

If we let g_0 and g_1 act on G/H we get the permutations
$\sigma_0 = (1\,3\,7\,4)(2\,6\,5\,8)(9)(10\,12)(11)$ and $\sigma_1 = (1\,7\,3\,11\,4)(2\,9\,5\,8\,6)(10)(12)$.

If we let g_0 and g_1 act on G/H' we get the permutations
$\sigma_0' = (1\,3\,10\,4)(2\,7\,12\,9)(6)(8)(5\,11)$ and $\sigma_1' = (1\,5\,10\,7\,9)(2\,4\,3\,12\,11)(6)(8)$.

The two resulting dessins are the following:

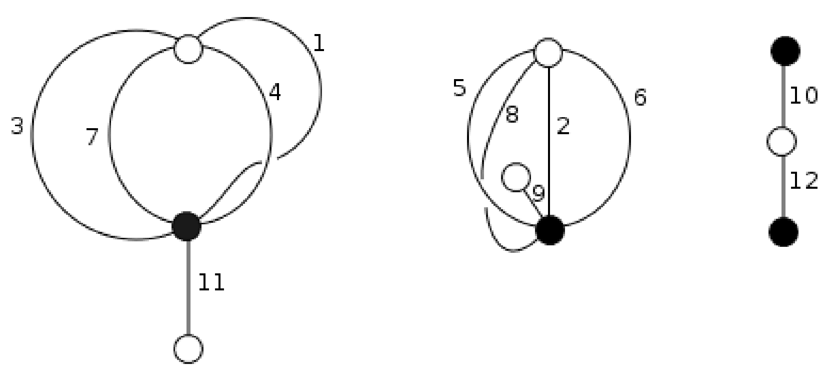

FIGURE 4. $\mathbb{D}(G/H, g_0, g_1)$
$\gamma_1 + \gamma_2 + \gamma_3 = 1 + 1 + 0 = 2$.

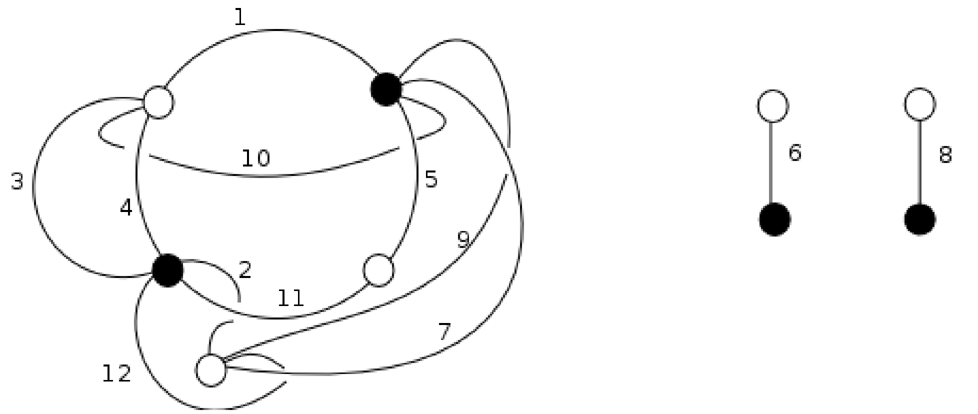

FIGURE 5. $\mathbb{D}(G/H', g_0, g_1)$
$\gamma'_1 + \gamma'_2 + \gamma'_3 = 1 + 1 + 0 = 2.$